Contents

Preface *xv*

1 Vector and tensor analysis 1
Vectors and scalars 1
Direction angles and direction cosines 3
Vector algebra 4
Equality of vectors 4
Vector addition 4
Multiplication by a scalar 4
The scalar product 5
The vector (cross or outer) product 7
The triple scalar product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ 10
The triple vector product 11
Change of coordinate system 11
The linear vector space V_n 13
Vector differentiation 15
Space curves 16
Motion in a plane 17
A vector treatment of classical orbit theory 18
Vector differential of a scalar field and the gradient 20
Conservative vector field 21
The vector differential operator ∇ 22
Vector differentiation of a vector field 22
The divergence of a vector 22
The operator ∇^2, the Laplacian 24
The curl of a vector 24
Formulas involving ∇ 27
Orthogonal curvilinear coordinates 27

Published online by Cambridge University Press

Special orthogonal coordinate systems 32
Cylindrical coordinates (ρ, ϕ, z) 32
Spherical coordinates (r, θ, ϕ) 34
Vector integration and integral theorems 35
Gauss' theorem (the divergence theorem) 37
Continuity equation 39
Stokes' theorem 40
Green's theorem 43
Green's theorem in the plane 44
Helmholtz's theorem 44
Some useful integral relations 45
Tensor analysis 47
Contravariant and covariant vectors 48
Tensors of second rank 48
Basic operations with tensors 49
Quotient law 50
The line element and metric tensor 51
Associated tensors 53
Geodesics in a Riemannian space 53
Covariant differentiation 55

Problems 57

2 Ordinary differential equations 62
First-order differential equations 63
Separable variables 63
Exact equations 67
Integrating factors 69
Bernoulli's equation 72
Second-order equations with constant coefficients 72
Nature of the solution of linear equations 73
General solutions of the second-order equations 74
Finding the complementary function 74
Finding the particular integral 77
Particular integral and the operator $D(= d/dx)$ 78
Rules for D operators 79
The Euler linear equation 83
Solutions in power series 85
Ordinary and singular points of a differential equation 86
Frobenius and Fuchs theorem 86
Simultaneous equations 93
The gamma and beta functions 94

Problems 96

Published online by Cambridge University Press

3 Matrix algebra 100
Definition of a matrix 100
Four basic algebra operations for matrices 102
Equality of matrices 102
Addition of matrices 102
Multiplication of a matrix by a number 103
Matrix multiplication 103
The commutator 107
Powers of a matrix 107
Functions of matrices 107
Transpose of a matrix 108
Symmetric and skew-symmetric matrices 109
The matrix representation of a vector product 110
The inverse of a matrix 111
A method for finding $\tilde{A}^{-1}$ 112
Systems of linear equations and the inverse of a matrix 113
Complex conjugate of a matrix 114
Hermitian conjugation 114
Hermitian/anti-hermitian matrix 114
Orthogonal matrix (real) 115
Unitary matrix 116
Rotation matrices 117
Trace of a matrix 121
Orthogonal and unitary transformations 121
Similarity transformation 122
The matrix eigenvalue problem 124
Determination of eigenvalues and eigenvectors 124
Eigenvalues and eigenvectors of hermitian matrices 128
Diagonalization of a matrix 129
Eigenvectors of commuting matrices 133
Cayley–Hamilton theorem 134
Moment of inertia matrix 135
Normal modes of vibrations 136
Direct product of matrices 139

Problems 140

4 Fourier series and integrals 144
Periodic functions 144
Fourier series; Euler–Fourier formulas 146
Gibb's phenomena 150
Convergence of Fourier series and Dirichlet conditions 150

Published online by Cambridge University Press

Half-range Fourier series 151
Change of interval 152
Parseval's identity 153
Alternative forms of Fourier series 155
Integration and differentiation of a Fourier series 157
Vibrating strings 157
The equation of motion of transverse vibration 157
Solution of the wave equation 158
RLC circuit 160
Orthogonal functions 162
Multiple Fourier series 163
Fourier integrals and Fourier transforms 164
Fourier sine and cosine transforms 172
Heisenberg's uncertainty principle 173
Wave packets and group velocity 174
Heat conduction 179
Heat conduction equation 179
Fourier transforms for functions of several variables 182
The Fourier integral and the delta function 183
Parseval's identity for Fourier integrals 186
The convolution theorem for Fourier transforms 188
Calculations of Fourier transforms 190
The delta function and Green's function method 192

Problems 195

5 Linear vector spaces 199
Euclidean n-space E_n 199
General linear vector spaces 201
Subspaces 203
Linear combination 204
Linear independence, bases, and dimensionality 204
Inner product spaces (unitary spaces) 206
The Gram–Schmidt orthogonalization process 209
The Cauchy–Schwarz inequality 210
Dual vectors and dual spaces 211
Linear operators 212
Matrix representation of operators 214
The algebra of linear operators 215
Eigenvalues and eigenvectors of an operator 217
Some special operators 217
The inverse of an operator 218

Published online by Cambridge University Press

The adjoint operators 219
Hermitian operators 220
Unitary operators 221
The projection operators 222
Change of basis 224
Commuting operators 225
Function spaces 226

Problems 230

6 Functions of a complex variable 233
Complex numbers 233
Basic operations with complex numbers 234
Polar form of complex number 234
De Moivre's theorem and roots of complex numbers 237
Functions of a complex variable 238
Mapping 239
Branch lines and Riemann surfaces 240
The differential calculus of functions of a complex variable 241
Limits and continuity 241
Derivatives and analytic functions 243
The Cauchy–Riemann conditions 244
Harmonic functions 247
Singular points 248
Elementary functions of z 249
The exponential functions e^z (or $\exp(z)$) 249
Trigonometric and hyperbolic functions 251
The logarithmic functions $w = \ln z$ 252
Hyperbolic functions 253
Complex integration 254
Line integrals in the complex plane 254
Cauchy's integral theorem 257
Cauchy's integral formulas 260
Cauchy's integral formulas for higher derivatives 262
Series representations of analytic functions 265
Complex sequences 265
Complex series 266
Ratio test 268
Uniform covergence and the Weierstrass M-test 268
Power series and Taylor series 269
Taylor series of elementary functions 272
Laurent series 274

Published online by Cambridge University Press

Integration by the method of residues 279
Residues 279
The residue theorem 282
Evaluation of real definite integrals 283
Improper integrals of the rational function $\int_{-\infty}^{\infty} f(x)dx$ 283
Integrals of the rational functions of $\sin\theta$ and $\cos\theta$ $\int_0^{2\pi} G(\sin\theta, \cos\theta)d\theta$ 286
Fourier integrals of the form $\int_{-\infty}^{\infty} f(x)\begin{Bmatrix}\sin mx\\ \cos mx\end{Bmatrix}dx$ 288
Problems 292

7 Special functions of mathematical physics 296
Legendre's equation 296
Rodrigues' formula for $P_n(x)$ 299
The generating function for $P_n(x)$ 301
Orthogonality of Legendre polynomials 304
The associated Legendre functions 307
Orthogonality of associated Legendre functions 309
Hermite's equation 311
Rodrigues' formula for Hermite polynomials $H_n(x)$ 313
Recurrence relations for Hermite polynomials 313
Generating function for the $H_n(x)$ 314
The orthogonal Hermite functions 314
Laguerre's equation 316
The generating function for the Laguerre polynomials $L_n(x)$ 317
Rodrigues' formula for the Laguerre polynomials $L_n(x)$ 318
The orthogonal Laugerre functions 319
The associated Laguerre polynomials $L_n^m(x)$ 320
Generating function for the associated Laguerre polynomials 320
Associated Laguerre function of integral order 321
Bessel's equation 321
Bessel functions of the second kind $Y_n(x)$ 325
Hanging flexible chain 328
Generating function for $J_n(x)$ 330
Bessel's integral representation 331
Recurrence formulas for $J_n(x)$ 332
Approximations to the Bessel functions 335
Orthogonality of Bessel functions 336
Spherical Bessel functions 338

Published online by Cambridge University Press

Sturm–Liouville systems 340

Problems 343

8 The calculus of variations 347
The Euler–Lagrange equation 348
Variational problems with constraints 353
Hamilton's principle and Lagrange's equation of motion 355
Rayleigh–Ritz method 359
Hamilton's principle and canonical equations of motion 361
The modified Hamilton's principle and the Hamilton–Jacobi equation 364
Variational problems with several independent variables 367

Problems 369

9 The Laplace transformation 372
Definition of the Lapace transform 372
Existence of Laplace transforms 373
Laplace transforms of some elementary functions 375
Shifting (or translation) theorems 378
The first shifting theorem 378
The second shifting theorem 379
The unit step function 380
Laplace transform of a periodic function 381
Laplace transforms of derivatives 382
Laplace transforms of functions defined by integrals 383
A note on integral transformations 384

Problems 385

10 Partial differential equations 387
Linear second-order partial differential equations 388
Solutions of Laplace's equation: separation of variables 392
Solutions of the wave equation: separation of variables 402
Solution of Poisson's equation. Green's functions 404
Laplace transform solutions of boundary-value problems 409

Problems 410

11 Simple linear integral equations 413
Classification of linear integral equations 413
Some methods of solution 414
Separable kernel 414
Neumann series solutions 416

Published online by Cambridge University Press

Transformation of an integral equation into a differential equation 419
Laplace transform solution 420
Fourier transform solution 421
The Schmidt–Hilbert method of solution 421
Relation between differential and integral equations 425
Use of integral equations 426
Abel's integral equation 426
Classical simple harmonic oscillator 427
Quantum simple harmonic oscillator 427

Problems 428

12 Elements of group theory 430
Definition of a group (group axioms) 430
Cyclic groups 433
Group multiplication table 434
Isomorphic groups 435
Group of permutations and Cayley's theorem 438
Subgroups and cosets 439
Conjugate classes and invariant subgroups 440
Group representations 442
Some special groups 444
The symmetry group D_2, D_3 446
One-dimensional unitary group $U(1)$ 449
Orthogonal groups $SO(2)$ and $SO(3)$ 450
The $SU(n)$ groups 452
Homogeneous Lorentz group 454

Problems 457

13 Numerical methods 459
Interpolation 459
Finding roots of equations 460
Graphical methods 460
Method of linear interpolation (method of false position) 461
Newton's method 464
Numerical integration 466
The rectangular rule 466
The trapezoidal rule 467
Simpson's rule 469
Numerical solutions of differential equations 469
Euler's method 470
The three-term Taylor series method 472

Published online by Cambridge University Press

The Runge–Kutta method 473
Equations of higher order. System of equations 476
Least-squares fit 477
Problems 478

14 Introduction to probability theory 481
A definition of probability 481
Sample space 482
Methods of counting 484
Permutations 484
Combinations 485
Fundamental probability theorems 486
Random variables and probability distributions 489
Random variables 489
Probability distributions 489
Expectation and variance 490
Special probability distributions 491
The binomial distribution 491
The Poisson distribution 495
The Gaussian (or normal) distribution 497
Continuous distributions 500
The Gaussian (or normal) distribution 502
The Maxwell–Boltzmann distribution 503

Problems 503

Appendix 1 Preliminaries (review of fundamental concepts) **506**
Inequalities 507
Functions 508
Limits 510
Infinite series 511
Tests for convergence 513
Alternating series test 516
Absolute and conditional convergence 517
Series of functions and uniform convergence 520
Weistrass *M* test 521
Abel's test 522
Theorem on power series 524
Taylor's expansion 524
Higher derivatives and Leibnitz's formula for *n*th derivative of a product 528
Some important properties of definite integrals 529

Published online by Cambridge University Press

Some useful methods of integration 531
Reduction formula 533
Differentiation of integrals 534
Homogeneous functions 535
Taylor series for functions of two independent variables 535
Lagrange multiplier 536

Appendix 2 Determinants 538
Determinants, minors, and cofactors 540
Expansion of determinants 541
Properties of determinants 542
Derivative of a determinant 547

Appendix 3 Table of function $F(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$ **548**

Further reading 549
Index 551

Published online by Cambridge University Press