# 1

# Vector and tensor analysis

## Vectors and scalars

Vector methods have become standard tools for the physicists. In this chapter we discuss the properties of the vectors and vector fields that occur in classical physics. We will do so in a way, and in a notation, that leads to the formation of abstract linear vector spaces in Chapter 5.

A physical quantity that is completely specified, in appropriate units, by a single number (called its magnitude) such as volume, mass, and temperature is called a scalar. Scalar quantities are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on.

There are also physical quantities which require a magnitude and a direction for their complete specification. These are called vectors *if* their combination with each other is commutative (that is the order of addition may be changed without affecting the result). Thus not all quantities possessing magnitude and direction are vectors. Angular displacement, for example, may be characterised by magnitude and direction but is not a vector, for the addition of two or more angular displacements is not, in general, commutative (Fig. 1.1).

In print, we shall denote vectors by boldface letters (such as **A**) and use ordinary italic letters (such as $A$) for their magnitudes; in writing, vectors are usually represented by a letter with an arrow above it such as $\vec{A}$. A given vector **A** (or $\vec{A}$) can be written as

$$\mathbf{A} = A\hat{A}, \tag{1.1}$$

where $A$ is the magnitude of vector **A** and so it has unit and dimension, and $\hat{A}$ is a dimensionless unit vector with a unity magnitude having the direction of **A**. Thus $\hat{A} = \mathbf{A}/A$.

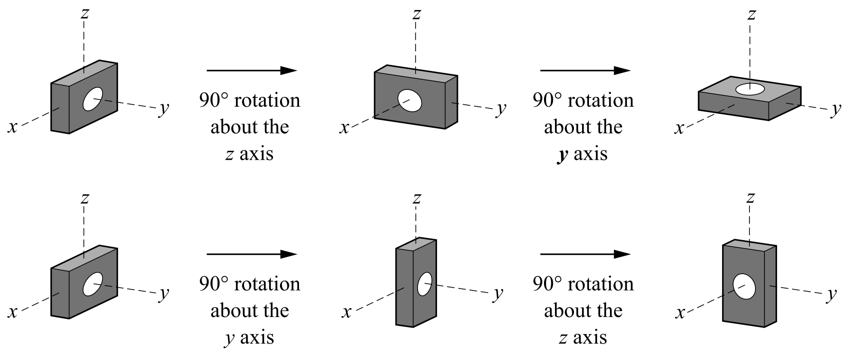


Figure 1.1. Rotation of a parallelpiped about coordinate axes.

A vector quantity may be represented graphically by an arrow-tipped line segment. The length of the arrow represents the magnitude of the vector, and the direction of the arrow is that of the vector, as shown in Fig. 1.2. Alternatively, a vector can be specified by its components (projections along the coordinate axes) and the unit vectors along the coordinate axes (Fig. 1.3):

$$\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i, \tag{1.2}$$

where $\hat{e}_i$ $(i = 1, 2, 3)$ are unit vectors along the rectangular axes $x_i$ $(x_1 = x, x_2 = y, x_3 = z)$; they are normally written as $\hat{i}, \hat{j}, \hat{k}$ in general physics textbooks. The component triplet $(A_1, A_2, A_3)$ is also often used as an alternate designation for vector $\mathbf{A}$:

$$\mathbf{A} = (A_1, A_2, A_3). \tag{1.2a}$$

This algebraic notation of a vector can be extended (or generalized) to spaces of dimension greater than three, where an ordered $n$-tuple of real numbers, $(A_1, A_2, \ldots, A_n)$, represents a vector. Even though we cannot construct physical vectors for $n > 3$, we can retain the geometrical language for these $n$-dimensional generalizations. Such abstract "vectors" will be the subject of Chapter 5.

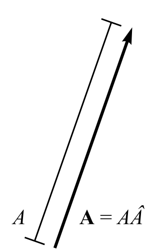


Figure 1.2. Graphical representation of vector **A**.



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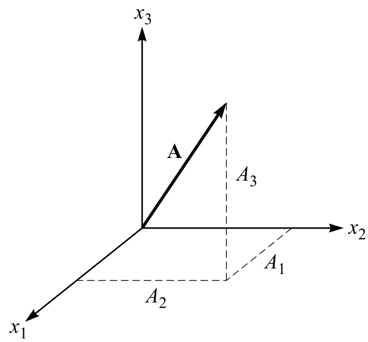


Figure 1.3. A vector **A** in Cartesian coordinates.

### Direction angles and direction cosines

We can express the unit vector $\hat{A}$ in terms of the unit coordinate vectors $\hat{e}_i$. From Eq. (1.2), $\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3$, we have

$$\mathbf{A} = A\left(\frac{A_1}{A}\hat{e}_1 + \frac{A_2}{A}\hat{e}_2 + \frac{A_3}{A}\hat{e}_3\right) = A\hat{A}.$$

Now $A_1/A = \cos\alpha, A_2/A = \cos\beta$, and $A_3/A = \cos\gamma$ are the direction cosines of the vector **A**, and $\alpha$, $\beta$, and $\gamma$ are the direction angles (Fig. 1.4). Thus we can write

$$\mathbf{A} = A(\cos\alpha\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = A\hat{A};$$

it follows that

$$\hat{A} = (\cos\alpha\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = (\cos\alpha, \cos\beta, \cos\gamma). \tag{1.3}$$

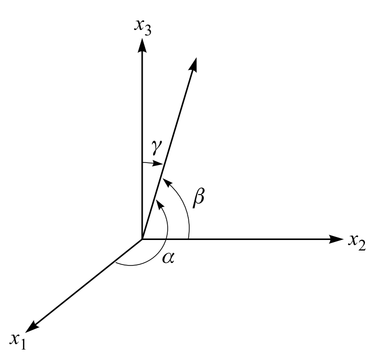


Figure 1.4. Direction angles of vector **A**.



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## Vector algebra

### *Equality of vectors*

Two vectors, say **A** and **B**, are equal if, and only if, their respective components are equal:

$$\mathbf{A} = \mathbf{B} \quad \text{or} \quad (A_1, A_2, A_3) = (B_1, B_2, B_3)$$

is equivalent to the three equations

$$A_1 = B_1, A_2 = B_2, A_3 = B_3.$$

Geometrically, equal vectors are parallel and have the same length, but do not necessarily have the same position.

### *Vector addition*

The addition of two vectors is defined by the equation

$$\mathbf{A} + \mathbf{B} = (A_1, A_2, A_3) + (B_1, B_2, B_3) = (A_1 + B_1, A_2 + B_2, A_3 + B_3).$$

That is, the sum of two vectors is a vector whose components are sums of the components of the two given vectors.

We can add two non-parallel vectors by graphical method as shown in Fig. 1.5. To add vector **B** to vector **A**, shift **B** parallel to itself until its tail is at the head of **A**. The vector sum $\mathbf{A} + \mathbf{B}$ is a vector **C** drawn from the tail of **A** to the head of **B**. The order in which the vectors are added does not affect the result.

### *Multiplication by a scalar*

If $c$ is scalar then

$$c\mathbf{A} = (cA_1, cA_2, cA_3).$$

Geometrically, the vector $c\mathbf{A}$ is parallel to **A** and is $c$ times the length of **A**. When $c = -1$, the vector $-\mathbf{A}$ is one whose direction is the reverse of that of **A**, but both

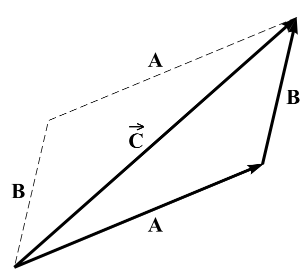


Figure 1.5. Addition of two vectors.



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have the same length. Thus, subtraction of vector **B** from vector **A** is equivalent to adding $-\mathbf{B}$ to **A**:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

We see that vector addition has the following properties:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutativity);
(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (associativity);
(c) $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$;
(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

We now turn to vector multiplication. Note that division by a vector is not defined: expressions such as $k/\mathbf{A}$ or $\mathbf{B}/\mathbf{A}$ are meaningless.

There are several ways of multiplying two vectors, each of which has a special meaning; two types are defined.

## The scalar product

The scalar (dot or inner) product of two vectors **A** and **B** is a real number defined (in geometrical language) as the product of their magnitude and the cosine of the (smaller) angle between them (Figure 1.6):

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta \qquad (0 \le \theta \le \pi). \tag{1.4}$$

It is clear from the definition (1.4) that the scalar product is commutative:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \tag{1.5}$$

and the product of a vector with itself gives the square of the dot product of the vector:

$$\mathbf{A} \cdot \mathbf{A} = A^2. \tag{1.6}$$

If $\mathbf{A} \cdot \mathbf{B} = 0$ and neither **A** nor **B** is a null (zero) vector, then **A** is perpendicular to **B**.

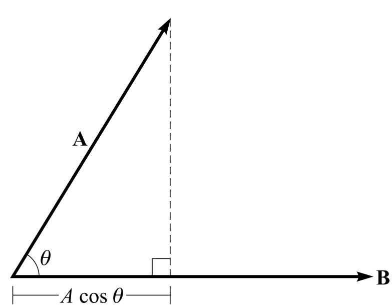


Figure 1.6. The scalar product of two vectors.



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We can get a simple geometric interpretation of the dot product from an inspection of Fig. 1.6:

$$(B\cos\theta)A = \text{projection of } \mathbf{B} \text{ onto } \mathbf{A} \text{ multiplied by the magnitude of } \mathbf{A},$$

$$(A\cos\theta)B = \text{projection of } \mathbf{A} \text{ onto } \mathbf{B} \text{ multiplied by the magnitude of } \mathbf{B}.$$

If only the components of **A** and **B** are known, then it would not be practical to calculate $\mathbf{A}\cdot\mathbf{B}$ from definition (1.4). But, in this case, we can calculate $\mathbf{A}\cdot\mathbf{B}$ in terms of the components:

$$\mathbf{A}\cdot\mathbf{B} = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)\cdot(B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3); \tag{1.7}$$

the right hand side has nine terms, all involving the product $\hat{e}_i\cdot\hat{e}_j$ . Fortunately, the angle between each pair of unit vectors is 90°, and from (1.4) and (1.6) we find that

$$\hat{e}_i\cdot\hat{e}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{1.8}$$

where $\delta_{ij}$ is the Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \tag{1.9}$$

After we use (1.8) to simplify the resulting nine terms on the right-side of (7), we obtain

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i. \tag{1.10}$$

The law of cosines for plane triangles can be easily proved with the application of the scalar product: refer to Fig. 1.7, where **C** is the resultant vector of **A** and **B**. Taking the dot product of **C** with itself, we obtain

$$\begin{aligned} C^2 &= \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) \\ &= A^2 + B^2 + 2\mathbf{A}\cdot\mathbf{B} = A^2 + B^2 + 2AB\cos\theta, \end{aligned}$$

which is the law of cosines.

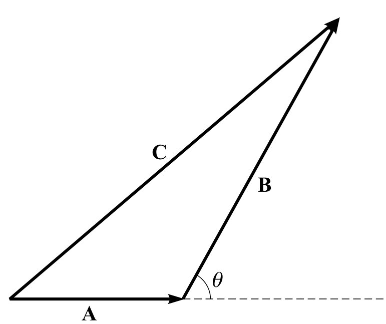


Figure 1.7. Law of cosines.



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A simple application of the scalar product in physics is the work $W$ done by a constant force $\mathbf{F}$: $W = \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{r}$ is the displacement vector of the object moved by $\mathbf{F}$.

### The vector (cross or outer) product

The vector product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector and is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}. \tag{1.11}$$

As shown in Fig. 1.8, the two vectors $\mathbf{A}$ and $\mathbf{B}$ form two sides of a parallelogram. We define $\mathbf{C}$ to be perpendicular to the plane of this parallelogram with its magnitude equal to the area of the parallelogram. And we choose the direction of $\mathbf{C}$ along the thumb of the right hand when the fingers rotate from $\mathbf{A}$ to $\mathbf{B}$ (angle of rotation less than 180°).

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\theta \hat{e}_C \qquad (0 \leq \theta \leq \pi). \tag{1.12}$$

From the definition of the vector product and following the right hand rule, we can see immediately that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1.13}$$

Hence the vector product is not commutative. If $\mathbf{A}$ and $\mathbf{B}$ are parallel, then it follows from Eq. (1.12) that

$$\mathbf{A} \times \mathbf{B} = 0. \tag{1.14}$$

In particular

$$\mathbf{A} \times \mathbf{A} = 0. \tag{1.14a}$$

In vector components, we have

$$\mathbf{A} \times \mathbf{B} = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \times (B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3). \tag{1.15}$$

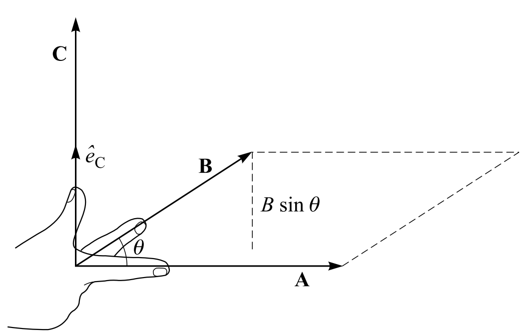


Figure 1.8. The right hand rule for vector product.



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Using the following relations

$$\begin{aligned} &\hat{e}_i \times \hat{e}_i = 0, \; i = 1, 2, 3, \\ &\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \; \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \; \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \end{aligned} \tag{1.16}$$

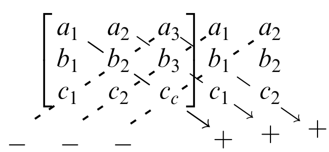
Eq. (1.15) becomes

$$\mathbf{A} \times \mathbf{B} = (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2 + (A_1 B_2 - A_2 B_1)\hat{e}_3. \tag{1.15a}$$

This can be written as an easily remembered determinant of third order:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}. \tag{1.17}$$

The expansion of a determinant of third order can be obtained by diagonal multiplication by repeating on the right the first two columns of the determinant and adding the signed products of the elements on the various diagonals in the resulting array:




The non-commutativity of the vector product of two vectors now appears as a consequence of the fact that interchanging two rows of a determinant changes its sign, and the vanishing of the vector product of two vectors in the same direction appears as a consequence of the fact that a determinant vanishes if one of its rows is a multiple of another.

The determinant is a basic tool used in physics and engineering. The reader is assumed to be familiar with this subject. Those who are in need of review should read Appendix II.

The vector resulting from the vector product of two vectors is called an axial vector, while ordinary vectors are sometimes called polar vectors. Thus, in Eq. (1.11), **C** is a pseudovector, while **A** and **B** are axial vectors. On an inversion of coordinates, polar vectors change sign but an axial vector does not change sign.

A simple application of the vector product in physics is the torque $\boldsymbol{\tau}$ of a force **F** about a point $O$: $\boldsymbol{\tau} = \mathbf{F} \times \mathbf{r}$, where **r** is the vector from $O$ to the initial point of the force **F** (Fig. 1.9).

We can write the nine equations implied by Eq. (1.16) in terms of permutation symbols $\varepsilon_{ijk}$:

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk} \hat{e}_k, \tag{1.16a}$$

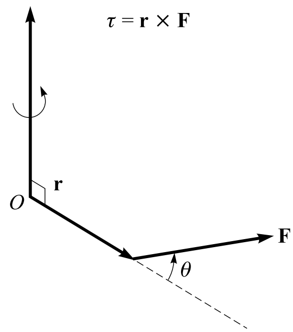



Figure 1.9. The torque of a force about a point $O$.

where $\varepsilon_{ijk}$ is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise (for example, if 2 or more indices are equal).} \end{cases} \tag{1.18}$$

It follows immediately that

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}.$$

There is a very useful identity relating the $\varepsilon_{ijk}$ and the Kronecker delta symbol:

$$\sum_{k=1}^{3} \varepsilon_{mnk}\varepsilon_{ijk} = \delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}, \tag{1.19}$$

$$\sum_{j,k} \varepsilon_{mjk}\varepsilon_{njk} = 2\delta_{mn}, \quad \sum_{i,j,k} \varepsilon_{ijk}^2 = 6. \tag{1.19a}$$

Using permutation symbols, we can now write the vector product $\mathbf{A} \times \mathbf{B}$ as

$$\mathbf{A} \times \mathbf{B} = \left(\sum_{i=1}^{3} A_i\hat{e}_i\right) \times \left(\sum_{j=1}^{3} B_j\hat{e}_j\right) = \sum_{i,j}^{3} A_iB_j(\hat{e}_i \times \hat{e}_j) = \sum_{i,j,k}^{3} (A_iB_j\varepsilon_{ijk})\hat{e}_k.$$

Thus the $k$th component of $\mathbf{A} \times \mathbf{B}$ is

$$(\mathbf{A} \times \mathbf{B})_k = \sum_{i,j} A_iB_j\varepsilon_{ijk} = \sum_{i,j} \varepsilon_{kij}A_iB_j.$$

If $k = 1$, we obtain the usual geometrical result:

$$(\mathbf{A} \times \mathbf{B})_1 = \sum_{i,j} \varepsilon_{1ij}A_iB_j = \varepsilon_{123}A_2B_3 + \varepsilon_{132}A_3B_2 = A_2B_3 - A_3B_2.$$



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### The triple scalar product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

We now briefly discuss the scalar $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. This scalar represents the volume of the parallelepiped formed by the coterminous sides $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, since

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = ABC \sin\theta \cos\alpha = hS = \text{volume},$$

$S$ being the area of the parallelogram with sides $\mathbf{B}$ and $\mathbf{C}$, and $h$ the height of the parallelogram (Fig. 1.10).

Now

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \cdot \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

so that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}. \tag{1.20}$$

The exchange of two rows (or two columns) changes the sign of the determinant but does not change its absolute value. Using this property, we find

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

that is, the dot and the cross may be interchanged in the triple scalar product.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{1.21}$$

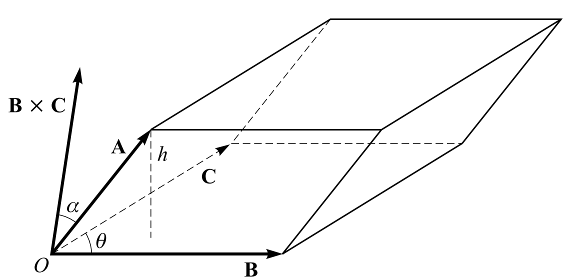


Figure 1.10. The triple scalar product of three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

In fact, as long as the three vectors appear in cyclic order, $\mathbf{A} \to \mathbf{B} \to \mathbf{C} \to \mathbf{A}$, then the dot and cross may be inserted between any pairs:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$

It should be noted that the scalar resulting from the triple scalar product changes sign on an inversion of coordinates. For this reason, the triple scalar product is sometimes called a pseudoscalar.

## The triple vector product

The triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a vector, since it is the vector product of two vectors: $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$. This vector is perpendicular to $\mathbf{B} \times \mathbf{C}$ and so it lies in the plane of $\mathbf{B}$ and $\mathbf{C}$. If $\mathbf{B}$ is not parallel to $\mathbf{C}$, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = x\mathbf{B} + y\mathbf{C}$. Now dot both sides with $\mathbf{A}$ and we obtain $x(\mathbf{A} \cdot \mathbf{B}) + y(\mathbf{A} \cdot \mathbf{C}) = 0$, since $\mathbf{A} \cdot [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] = 0$. Thus

$$x/(\mathbf{A} \cdot \mathbf{C}) = -y/(\mathbf{A} \cdot \mathbf{B}) \equiv \lambda \quad (\lambda \text{ is a scalar})$$

and so

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = x\mathbf{B} + y\mathbf{C} = \lambda[\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})].$$

We now show that $\lambda = 1$. To do this, let us consider the special case when $\mathbf{B} = \mathbf{A}$. Dot the last equation with $\mathbf{C}$:

$$\mathbf{C} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = \lambda[(\mathbf{A} \cdot \mathbf{C})^2 - \mathbf{A}^2\mathbf{C}^2],$$

or, by an interchange of dot and cross

$$-(\mathbf{A} \cdot \mathbf{C})^2 = \lambda[(\mathbf{A} \cdot \mathbf{C})^2 - \mathbf{A}^2\mathbf{C}^2].$$

In terms of the angles between the vectors and their magnitudes the last equation becomes

$$-A^2C^2 \sin^2\theta = \lambda(A^2C^2 \cos^2\theta - A^2C^2) = -\lambda A^2C^2 \sin^2\theta;$$

hence $\lambda = 1$. And so

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{1.22}$$

## Change of coordinate system

Vector equations are independent of the coordinate system we happen to use. But the components of a vector quantity are different in different coordinate systems. We now make a brief study of how to represent a vector in different coordinate systems. As the rectangular Cartesian coordinate system is the basic type of coordinate system, we shall limit our discussion to it. Other coordinate systems



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will be introduced later. Consider the vector **A** expressed in terms of the unit coordinate vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$:

$$\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3 = \sum_{i=1}^{3} A_i\hat{e}_i.$$

Relative to a new system $(\hat{e}_1', \hat{e}_2', \hat{e}_3')$ that has a different orientation from that of the old system $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, vector **A** is expressed as

$$\mathbf{A} = A_1'\hat{e}_1' + A_2'\hat{e}_2' + A'\hat{e}_3' = \sum_{i=1}^{3} A_i'\hat{e}_i'.$$

Note that the dot product $\mathbf{A} \cdot \hat{e}_1'$ is equal to $A_1'$, the projection of **A** on the direction of $\hat{e}_1'$; $\mathbf{A} \cdot \hat{e}_2'$ is equal to $A_2'$, and $\mathbf{A} \cdot \hat{e}_3'$ is equal to $A_3'$. Thus we may write

$$\left.\begin{aligned} A_1' &= (\hat{e}_1 \cdot \hat{e}_1')A_1 + (\hat{e}_2 \cdot \hat{e}_1')A_2 + (\hat{e}_3 \cdot \hat{e}_1')A_3, \\ A_2' &= (\hat{e}_1 \cdot \hat{e}_2')A_1 + (\hat{e}_2 \cdot \hat{e}_2')A_2 + (\hat{e}_3 \cdot \hat{e}_2')A_3, \\ A_3' &= (\hat{e}_1 \cdot \hat{e}_3')A_1 + (\hat{e}_2 \cdot \hat{e}_3')A_2 + (\hat{e}_3 \cdot \hat{e}_3')A_3. \end{aligned}\right\} \tag{1.23}$$

The dot products $(\hat{e}_i \cdot \hat{e}_j')$ are the direction cosines of the axes of the new coordinate system relative to the old system: $\hat{e}_i' \cdot \hat{e}_j = \cos(x_i', x_j)$; they are often called the coefficients of transformation. In matrix notation, we can write the above system of equations as

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \hat{e}_1 \cdot \hat{e}_1' & \hat{e}_2 \cdot \hat{e}_1' & \hat{e}_3 \cdot \hat{e}_1' \\ \hat{e}_1 \cdot \hat{e}_2' & \hat{e}_2 \cdot \hat{e}_2' & \hat{e}_3 \cdot \hat{e}_2' \\ \hat{e}_1 \cdot \hat{e}_3' & \hat{e}_2 \cdot \hat{e}_3' & \hat{e}_3 \cdot \hat{e}_3' \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

The $3 \times 3$ matrix in the above equation is called the rotation (or transformation) matrix, and is an orthogonal matrix. One advantage of using a matrix is that successive transformations can be handled easily by means of matrix multiplication. Let us digress for a quick review of some basic matrix algebra. A full account of matrix method is given in Chapter 3.

A matrix is an ordered array of scalars that obeys prescribed rules of addition and multiplication. A particular matrix element is specified by its row number followed by its column number. Thus $a_{ij}$ is the matrix element in the $i$th row and $j$th column. Alternative ways of representing matrix $\tilde{A}$ are $[a_{ij}]$ or the entire array

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$



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$\tilde{A}$ is an $n \times m$ matrix. A vector is represented in matrix form by writing its components as either a row or column array, such as

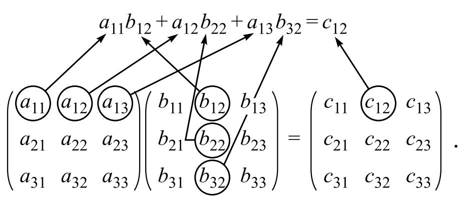
$$\tilde{B} = (b_{11}\ b_{12}\ b_{13}) \quad \text{or} \quad \tilde{C} = \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix},$$

where $b_{11} = b_x, b_{12} = b_y, b_{13} = b_z$, and $c_{11} = c_x, c_{21} = c_y, c_{31} = c_z$.

The multiplication of a matrix $\tilde{A}$ and a matrix $\tilde{B}$ is defined only when the number of columns of $\tilde{A}$ is equal to the number of rows of $\tilde{B}$, and is performed in the same way as the multiplication of two determinants: if $\tilde{C} = \tilde{A}\tilde{B}$, then

$$c_{ij} = \sum_k a_{ik} b_{kl}.$$

We illustrate the multiplication rule for the case of the $3 \times 3$ matrix $\tilde{A}$ multiplied by the $3 \times 3$ matrix $\tilde{B}$:




If we denote the direction cosines $\hat{e}'_i \cdot \hat{e}_j$ by $\lambda_{ij}$, then Eq. (1.23) can be written as

$$A'_i = \sum_{j=1}^{3} \hat{e}'_i \cdot \hat{e}_j A_j = \sum_{j=1}^{3} \lambda_{ij} A_j. \tag{1.23a}$$

It can be shown (Problem 1.9) that the quantities $\lambda_{ij}$ satisfy the following relations

$$\sum_{i=1}^{3} \lambda_{ij} \lambda_{ik} = \delta_{jk} \quad (j, k = 1, 2, 3). \tag{1.24}$$

Any linear transformation, such as Eq. (1.23a), that has the properties required by Eq. (1.24) is called an orthogonal transformation, and Eq. (1.24) is known as the orthogonal condition.

## The linear vector space $V_n$

We have found that it is very convenient to use vector components, in particular, the unit coordinate vectors $\hat{e}_i$ ($i = 1, 2, 3$). The three unit vectors $\hat{e}_i$ are orthogonal and normal, or, as we shall say, orthonormal. This orthonormal property is conveniently written as Eq. (1.8). But there is nothing special about these



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orthonormal unit vectors $\hat{e}_i$. If we refer the components of the vectors to a different system of rectangular coordinates, we need to introduce another set of three orthonormal unit vectors $\hat{f}_1, \hat{f}_2$, and $\hat{f}_3$:

$$\hat{f}_i \hat{f}_j = \delta_{ij} \quad (i, j = 1, 2, 3). \tag{1.8a}$$

For any vector **A** we now write

$$\mathbf{A} = \sum_{i=1}^{3} c_i \hat{f}_i, \quad \text{and} \quad c_i = \hat{f}_i \cdot \mathbf{A}.$$

We see that we can define a large number of different coordinate systems. But the physically significant quantities are the vectors themselves and certain functions of these, which are independent of the coordinate system used. The orthonormal condition (1.8) or (1.8a) is convenient in practice. If we also admit oblique Cartesian coordinates then the $\hat{f}_i$ need neither be normal nor orthogonal; they could be any three non-coplanar vectors, and any vector **A** can still be written as a linear superposition of the $\hat{f}_i$

$$\mathbf{A} = c_1 \hat{f}_1 + c_2 \hat{f}_2 + c_3 \hat{f}_3. \tag{1.25}$$

Starting with the vectors $\hat{f}_i$, we can find linear combinations of them by the algebraic operations of vector addition and multiplication of vectors by scalars, and then the collection of all such vectors makes up the three-dimensional linear space often called $V_3$ (V for vector) or $R_3$ ($R$ for real) or $E_3$ ($E$ for Euclidean). The vectors $\hat{f}_1, \hat{f}_2, \hat{f}_3$ are called the base vectors or bases of the vector space $V_3$. Any set of vectors, such as the $\hat{f}_i$, which can serve as the bases or base vectors of $V_3$ is called complete, and we say it spans the linear vector space. The base vectors are also linearly independent because no relation of the form

$$c_1 \hat{f}_1 + c_2 \hat{f}_2 + c_3 \hat{f}_3 = 0 \tag{1.26}$$

exists between them, unless $c_1 = c_2 = c_3 = 0$.

The notion of a vector space is much more general than the real vector space $V_3$. Extending the concept of $V_3$, it is convenient to call an ordered set of $n$ matrices, or functions, or operators, a 'vector' (or an $n$-vector) in the $n$-dimensional space $V_n$. Chapter 5 will provide justification for doing this. Taking a cue from $V_3$, vector addition in $V_n$ is defined to be

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \tag{1.27}$$

and multiplication by scalars is defined by

$$\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n), \tag{1.28}$$



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where $\alpha$ is real. With these two algebraic operations of vector addition and multiplication by scalars, we call $V_n$ a vector space. In addition to this algebraic structure, $V_n$ has geometric structure derived from the length defined to be

$$\left(\sum_{j=1}^{n} x_j^2\right)^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2} \tag{1.29}$$

The dot product of two $n$-vectors can be defined by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum_{j=1}^{n} x_j y_j. \tag{1.30}$$

In $V_n$, vectors are not directed line segments as in $V_3$; they may be an ordered set of $n$ operators, matrices, or functions. We do not want to become sidetracked from our main goal of this chapter, so we end our discussion of vector space here.

## Vector differentiation

Up to this point we have been concerned mainly with vector algebra. A vector may be a function of one or more scalars and vectors. We have encountered, for example, many important vectors in mechanics that are functions of time and position variables. We now turn to the study of the calculus of vectors.

Physicists like the concept of field and use it to represent a physical quantity that is a function of position in a given region. Temperature is a scalar field, because its value depends upon location: to each point ($x$, $y$, $z$) is associated a temperature $T(x, y, z)$. The function $T(x, y, z)$ is a scalar field, whose value is a real number depending only on the point in space but not on the particular choice of the coordinate system. A vector field, on the other hand, associates with each point a vector (that is, we associate three numbers at each point), such as the wind velocity or the strength of the electric or magnetic field. When described in a rotated system, for example, the three components of the vector associated with one and the same point will change in numerical value. Physically and geometrically important concepts in connection with scalar and vector fields are the gradient, divergence, curl, and the corresponding integral theorems.

The basic concepts of calculus, such as continuity and differentiability, can be naturally extended to vector calculus. Consider a vector $\mathbf{A}$, whose components are functions of a single variable $u$. If the vector $\mathbf{A}$ represents position or velocity, for example, then the parameter $u$ is usually time $t$, but it can be any quantity that determines the components of $\mathbf{A}$. If we introduce a Cartesian coordinate system, the vector function $\mathbf{A}(u)$ may be written as

$$\mathbf{A}(u) = A_1(u)\hat{e}_1 + A_2(u)\hat{e}_2 + A_3(u)\hat{e}_3. \tag{1.31}$$

$\mathbf{A}(u)$ is said to be continuous at $u = u_0$ if it is defined in some neighborhood of $u_0$ and

$$\lim_{u \to u_0} A(u) = A(u_0). \tag{1.32}$$

Note that $\mathbf{A}(u)$ is continuous at $u_0$ if and only if its three components are continuous at $u_0$.

$\mathbf{A}(u)$ is said to be differentiable at a point $u$ if the limit

$$\frac{d\mathbf{A}(u)}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u + \Delta u) - \mathbf{A}(u)}{\Delta u} \tag{1.33}$$

exists. The vector $\mathbf{A}'(u) = d\mathbf{A}(u)/du$ is called the derivative of $\mathbf{A}(u)$; and to differentiate a vector function we differentiate each component separately:

$$\mathbf{A}'(u) = A_1'(u)\hat{e}_1 + A_2'(u)\hat{e}_2 + A_3'(u)\hat{e}_3. \tag{1.33a}$$

Note that the unit coordinate vectors are fixed in space. Higher derivatives of $\mathbf{A}(u)$ can be similarly defined.

If $\mathbf{A}$ is a vector depending on more than one scalar variable, say $u$, $v$ for example, we write $\mathbf{A} = \mathbf{A}(u, v)$. Then

$$d\mathbf{A} = (\partial\mathbf{A}/\partial u)du + (\partial\mathbf{A}/\partial v)dv \tag{1.34}$$

is the differential of $\mathbf{A}$, and

$$\frac{\partial\mathbf{A}}{\partial u} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u + \Delta u, v) - \mathbf{A}(u, v)}{\partial u} \tag{1.34a}$$

and similarly for $\partial\mathbf{A}/\partial v$.

Derivatives of products obey rules similar to those for scalar functions. However, when cross products are involved the order may be important.

## Space curves

As an application of vector differentiation, let us consider some basic facts about curves in space. If $\mathbf{A}(u)$ is the position vector $\mathbf{r}(u)$ joining the origin of a coordinate system and any point $P(x_1, x_2, x_3)$ in space as shown in Fig. 1.11, then Eq. (1.31) becomes

$$\mathbf{r}(u) = x_1(u)\hat{e}_1 + x_2(u)\hat{e}_2 + x_3(u)\hat{e}_3. \tag{1.35}$$

As $u$ changes, the terminal point $P$ of $\mathbf{r}$ describes a curve $C$ in space. Eq. (1.35) is called a parametric representation of the curve $C$, and $u$ is the parameter of this representation. Then

$$\frac{\Delta\mathbf{r}}{\Delta u}\left(= \frac{\mathbf{r}(u + \Delta u) - \mathbf{r}(u)}{\Delta u}\right)$$

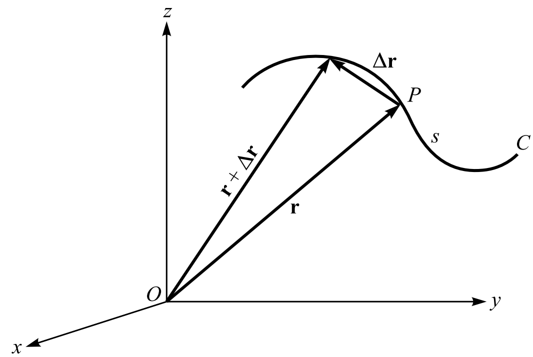


Figure 1.11. Parametric representation of a curve.

is a vector in the direction of $\Delta\mathbf{r}$, and its limit (if it exists) $d\mathbf{r}/du$ is a vector in the direction of the tangent to the curve at $(x_1, x_2, x_3)$. If $u$ is the arc length $s$ measured from some fixed point on the curve $C$, then $d\mathbf{r}/ds = \hat{T}$ is a unit tangent vector to the curve $C$. The rate at which $\hat{T}$ changes with respect to $s$ is a measure of the curvature of $C$ and is given by $d\hat{T}/ds$. The direction of $d\hat{T}/ds$ at any given point on $C$ is normal to the curve at that point: $\hat{T}\cdot\hat{T} = 1$, $d(\hat{T}\cdot\hat{T})/ds = 0$, from this we get $\hat{T}\cdot d\hat{T}/ds = 0$, so they are normal to each other. If $\hat{N}$ is a unit vector in this normal direction (called the principal normal to the curve), then $d\hat{T}/ds = \kappa\hat{N}$, and $\kappa$ is called the curvature of $C$ at the specified point. The quantity $\rho = 1/\kappa$ is called the radius of curvature. In physics, we often study the motion of particles along curves, so the above results may be of value.

In mechanics, the parameter $u$ is time $t$, then $d\mathbf{r}/dt = \mathbf{v}$ is the velocity of the particle which is tangent to the curve at the specific point. Now we can write

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\hat{T}$$

where $v$ is the magnitude of $\mathbf{v}$, called the speed. Similarly, $\mathbf{a} = d\mathbf{v}/dt$ is the acceleration of the particle.

## Motion in a plane

Consider a particle $P$ moving in a plane along a curve $C$ (Fig. 1.12). Now $\mathbf{r} = r\hat{e}_r$, where $\hat{e}_r$ is a unit vector in the direction of $\mathbf{r}$. Hence

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}.$$

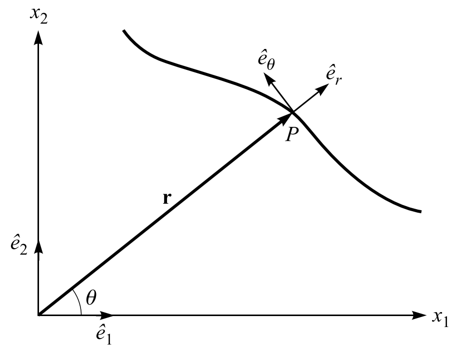

Figure 1.12. Motion in a plane.

Now $d\hat{e}_r/dt$ is perpendicular to $\hat{e}_r$. Also $|d\hat{e}_r/dt| = d\theta/dt$; we can easily verify this by differentiating $\hat{e}_r = \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2$. Hence

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta;$$

$\hat{e}_\theta$ is a unit vector perpendicular to $\hat{e}_r$.

Differentiating again we obtain

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta + r\frac{d\theta}{dt}\hat{e}_\theta$$

$$= \frac{d^2r}{dt^2}\hat{e}_r + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta - r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r \quad \left(\because \frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r\right).$$

Thus

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{e}_\theta.$$

**A vector treatment of classical orbit theory**

To illustrate the power and use of vector methods, we now employ them to work out the Keplerian orbits. We first prove Kepler's second law which can be stated as: angular momentum is constant in a central force field. A central force is a force whose line of action passes through a single point or center and whose magnitude depends only on the distance from the center. Gravity and electrostatic forces are central forces. A general discussion on central force can be found in, for example, Chapter 6 of *Classical Mechanics*, Tai L. Chow, John Wiley, New York, 1995.

Differentiating the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ with respect to time, we obtain

$$d\mathbf{L}/dt = d\mathbf{r}/dt \times \mathbf{p} + \mathbf{r} \times d\mathbf{p}/dt.$$

The first vector product vanishes because $\mathbf{p} = md\mathbf{r}/dt$ so $d\mathbf{r}/dt$ and $\mathbf{p}$ are parallel. The second vector product is simply $\mathbf{r} \times \mathbf{F}$ by Newton's second law, and hence vanishes for all forces directed along the position vector $\mathbf{r}$, that is, for all central forces. Thus the angular momentum $\mathbf{L}$ is a constant vector in central force motion. This implies that the position vector $\mathbf{r}$, and therefore the entire orbit, lies in a fixed plane in three-dimensional space. This result is essentially Kepler's second law, which is often stated in terms of the conservation of area velocity, $|\mathbf{L}|/2m$.

We now consider the inverse-square central force of gravitational and electrostatics. Newton's second law then gives

$$md\mathbf{v}/dt = -(k/r^2)\hat{n}, \tag{1.36}$$

where $\hat{n} = \mathbf{r}/r$ is a unit vector in the $\mathbf{r}$-direction, and $k = Gm_1m_2$ for the gravitational force, and $k = q_1q_2$ for the electrostatic force in cgs units. First we note that

$$\mathbf{v} = d\mathbf{r}/dt = dr/dt\hat{n} + rd\hat{n}/dt.$$

Then $\mathbf{L}$ becomes

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = mr^2[\hat{n} \times (d\hat{n}/dt)]. \tag{1.37}$$

Now consider

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{L}) = \frac{d\mathbf{v}}{dt} \times \mathbf{L} = -\frac{k}{mr^2}(\hat{n} \times \mathbf{L}) = -\frac{k}{mr^2}[\hat{n} \times mr^2(\hat{n} \times d\hat{n}/dt)]$$
$$= -k[\hat{n}(d\hat{n}/dt \cdot \hat{n}) - (d\hat{n}/dt)(\hat{n} \cdot \hat{n})].$$

Since $\hat{n} \cdot \hat{n} = 1$, it follows by differentiation that $\hat{n} \cdot d\hat{n}/dt = 0$. Thus we obtain

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{L}) = kd\hat{n}/dt;$$

integration gives

$$\mathbf{v} \times \mathbf{L} = k\hat{n} + \mathbf{C}, \tag{1.38}$$

where $\mathbf{C}$ is a constant vector. It lies along, and fixes the position of, the major axis of the orbit as we shall see after we complete the derivation of the orbit. To find the orbit, we form the scalar quantity

$$L^2 = \mathbf{L} \cdot (\mathbf{r} \times m\mathbf{v}) = m\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) = mr(k + C\cos\theta), \tag{1.39}$$

where $\theta$ is the angle measured from $\mathbf{C}$ (which we may take to be the $x$-axis) to $\mathbf{r}$. Solving for $r$, we obtain

$$r = \frac{L^2/km}{1 + C/(k\cos\theta)} = \frac{A}{1 + \varepsilon\cos\theta}. \tag{1.40}$$

Eq. (1.40) is a conic section with one focus at the origin, where $\varepsilon$ represents the eccentricity of the conic section; depending on its values, the conic section may be



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a circle, an ellipse, a parabola, or a hyperbola. The eccentricity can be easily determined in terms of the constants of motion:

$$\varepsilon = \frac{C}{k} = \frac{1}{k}|(\mathbf{v} \times \mathbf{L}) - k\hat{n}|$$
$$= \frac{1}{k}[|\mathbf{v} \times \mathbf{L}|^2 + k^2 - 2k\hat{n} \cdot (\mathbf{v} \times \mathbf{L})]^{1/2}$$

Now $|\mathbf{v} \times \mathbf{L}|^2 = v^2L^2$ because $\mathbf{v}$ is perpendicular to $\mathbf{L}$. Using Eq. (1.39), we obtain

$$\varepsilon = \frac{1}{k}\left[v^2L^2 + k^2 - \frac{2kL^2}{mr}\right]^{1/2} = \left[1 + \frac{2L^2}{mk^2}\left(\frac{1}{2}mv^2 - \frac{k}{r}\right)\right]^{1/2} = \left[1 + \frac{2L^2E}{mk^2}\right]^{1/2},$$

where $E$ is the constant energy of the system.

### Vector differentiation of a scalar field and the gradient

Given a scalar field in a certain region of space given by a scalar function $\phi(x_1, x_2, x_3)$ that is defined and differentiable at each point with respect to the position coordinates $(x_1, x_2, x_3)$, the total differential corresponding to an infinitesimal change $d\mathbf{r} = (dx_1, dx_2, dx_3)$ is

$$d\phi = \frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3. \tag{1.41}$$

We can express $d\phi$ as a scalar product of two vectors:

$$d\phi = \frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3 = (\nabla\phi) \cdot d\mathbf{r}, \tag{1.42}$$

where

$$\nabla\phi \equiv \frac{\partial \phi}{\partial x_1}\hat{e}_1 + \frac{\partial \phi}{\partial x_2}\hat{e}_2 + \frac{\partial \phi}{\partial x_3}\hat{e}_3 \tag{1.43}$$

is a vector field (or a vector point function). By this we mean to each point $\mathbf{r} = (x_1, x_2, x_3)$ in space we associate a vector $\nabla\phi$ as specified by its three components $(\partial\phi/\partial x_1, \partial\phi/\partial x_2, \partial\phi/\partial x_3)$: $\nabla\phi$ is called the *gradient* of $\phi$ and is often written as grad $\phi$.

There is a simple geometric interpretation of $\nabla\phi$. Note that $\phi(x_1, x_2, x_3) = c$, where $c$ is a constant, represents a surface. Let $\mathbf{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$ be the position vector to a point $P(x_1, x_2, x_3)$ on the surface. If we move along the surface to a nearby point $Q(\mathbf{r} + d\mathbf{r})$, then $d\mathbf{r} = dx_1\hat{e}_1 + dx_2\hat{e}_2 + dx_3\hat{e}_3$ lies in the tangent plane to the surface at $P$. But as long as we move along the surface $\phi$ has a constant value and $d\phi = 0$. Consequently from (1.41),

$$d\mathbf{r} \cdot \nabla\phi = 0. \tag{1.44}$$

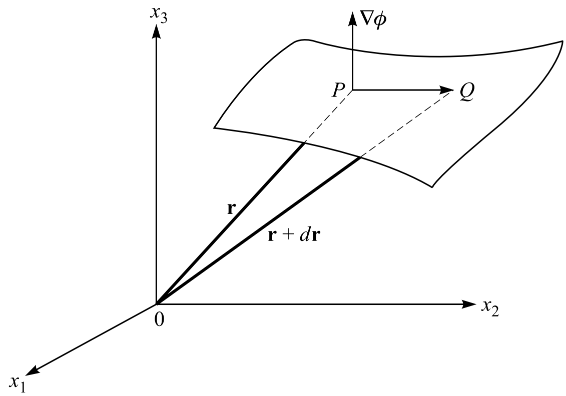


Figure 1.13. Gradient of a scalar.

Eq. (1.44) states that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface (Fig. 1.13). Let us return to

$$d\phi = (\nabla\phi) \cdot d\mathbf{r}.$$

The vector $\nabla\phi$ is fixed at any point $P$, so that $d\phi$, the change in $\phi$, will depend to a great extent on $d\mathbf{r}$. Consequently $d\phi$ will be a maximum when $d\mathbf{r}$ is parallel to $\nabla\phi$, since $d\mathbf{r} \cdot \nabla\phi = |d\mathbf{r}||\nabla\phi|\cos\theta$, and cos $\theta$ is a maximum for $\theta = 0$. Thus $\nabla\phi$ is in the direction of maximum increase of $\phi(x_1, x_2, x_3)$. The component of $\nabla\phi$ in the direction of a unit vector $\hat{u}$ is given by $\nabla\phi \cdot \hat{u}$ and is called the directional derivative of $\phi$ in the direction $\hat{u}$. Physically, this is the rate of change of $\phi$ at $(x_1, x_2, x_3)$ in the direction $\hat{u}$.

## Conservative vector field

By definition, a vector field is said to be conservative if the line integral of the vector along any closed path vanishes. Thus, if $\mathbf{F}$ is a conservative vector field (say, a conservative force field in mechanics), then

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0, \tag{1.45}$$

where $d\mathbf{s}$ is an element of the path. A (necessary and sufficient) condition for $\mathbf{F}$ to be conservative is that $\mathbf{F}$ can be expressed as the gradient of a scalar, say $\phi$: $\mathbf{F} = -\text{grad}\ \phi$:

$$\int_a^b \mathbf{F} \cdot d\mathbf{s} = -\int_a^b \text{grad}\,\phi \cdot d\mathbf{s} = -\int_a^b d\phi = \phi(a) - \phi(b):$$

it is obvious that the line integral depends solely on the value of the scalar $\phi$ at the initial and final points, and $\oint \mathbf{F} \cdot d\mathbf{s} = -\oint \text{grad}\,\phi \cdot d\mathbf{s} = 0$.

### The vector differential operator $\nabla$

We denoted the operation that changes a scalar field to a vector field in Eq. (1.43) by the symbol $\nabla$ (del or nabla):

$$\nabla \equiv \frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3, \tag{1.46}$$

which is called a gradient operator. We often write $\nabla\phi$ as grad $\phi$, and the vector field $\nabla\phi(\mathbf{r})$ is called the gradient of the scalar field $\phi(\mathbf{r})$. Notice that the operator $\nabla$ contains both partial differential operators and a direction: it is a vector differential operator. This important operator possesses properties analogous to those of ordinary vectors. It will help us in the future to keep in mind that $\nabla$ acts both as a differential operator and as a vector.

### Vector differentiation of a vector field

Vector differential operations on vector fields are more complicated because of the vector nature of both the operator and the field on which it operates. As we know there are two types of products involving two vectors, namely the scalar and vector products; vector differential operations on vector fields can also be separated into two types called the curl and the divergence.

#### *The divergence of a vector*

If $\mathbf{V}(x_1, x_2, x_3) = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$ is a differentiable vector field (that is, it is defined and differentiable at each point $(x_1, x_2, x_3)$ in a certain region of space), the divergence of $\mathbf{V}$, written $\nabla \cdot \mathbf{V}$ or div $\mathbf{V}$, is defined by the scalar product

$$\begin{aligned}\nabla \cdot \mathbf{V} &= \left(\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3\right) \cdot (V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3)\\ &= \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}.\end{aligned} \tag{1.47}$$

The result is a scalar field. Note the analogy with $\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$, but also note that $\nabla \cdot \mathbf{V} \neq \mathbf{V} \cdot \nabla$ (bear in mind that $\nabla$ is an operator). $\mathbf{V} \cdot \nabla$ is a scalar differential operator:

$$\mathbf{V} \cdot \nabla = V_1\frac{\partial}{\partial x_1} + V_2\frac{\partial}{\partial x_2} + V_3\frac{\partial}{\partial x_3}.$$

What is the physical significance of the divergence? Or why do we call the scalar product $\nabla \cdot \mathbf{V}$ the divergence of $\mathbf{V}$? To answer these questions, we consider, as an example, the steady motion of a fluid of density $\rho(x_1, x_2, x_3)$, and the velocity field is given by $\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)e_1 + v_2(x_1, x_2, x_3)e_2 + v_3(x_1, x_2, x_3)e_3$. We now concentrate on the flow passing through a small parallelepiped $ABCDEFGH$ of dimensions $dx_1 dx_2 dx_3$ (Fig. 1.14). The $x_1$ and $x_3$ components of the velocity $\mathbf{v}$ contribute nothing to the flow through the face $ABCD$. The mass of fluid entering $ABCD$ per unit time is given by $\rho v_2 dx_1 dx_3$ and the amount leaving the face $EFGH$ per unit time is

$$\left[\rho v_2 + \frac{\partial(\rho v_2)}{\partial x_2} dx_2\right] dx_1 dx_3.$$

So the loss of mass per unit time is $[\partial(\rho v_2)/\partial x_2] dx_1 dx_2 dx_3$. Adding the net rate of flow out all three pairs of surfaces of our parallelepiped, the total mass loss per unit time is

$$\left[\frac{\partial}{\partial x_1}(\rho v_1) + \frac{\partial}{\partial x_2}(\rho v_2) + \frac{\partial}{\partial x_3}(\rho v_3)\right] dx_1 dx_2 dx_3 = \nabla \cdot (\rho \mathbf{v}) dx_1 dx_2 dx_3.$$

So the mass loss per unit time per unit volume is $\nabla \cdot (\rho \mathbf{v})$. Hence the name divergence.

The divergence of any vector $\mathbf{V}$ is defined as $\nabla \cdot \mathbf{V}$. We now calculate $\nabla \cdot (f\mathbf{V})$, where $f$ is a scalar:

$$\begin{aligned}\nabla \cdot (f\,\mathbf{V}) &= \frac{\partial}{\partial x_1}(fV_1) + \frac{\partial}{\partial x_2}(fV_2) + \frac{\partial}{\partial x_3}(fV_3) \\ &= f\left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right) + \left(V_1 \frac{\partial f}{\partial x_1} + V_2 \frac{\partial f}{\partial x_2} + V_3 \frac{\partial f}{\partial x_3}\right)\end{aligned}$$

or

$$\nabla \cdot (f\mathbf{V}) = f \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla f. \tag{1.48}$$

It is easy to remember this result if we remember that $\nabla$ acts both as a differential operator and a vector. Thus, when operating on $f\,\mathbf{V}$, we first keep $f$ fixed and let $\nabla$

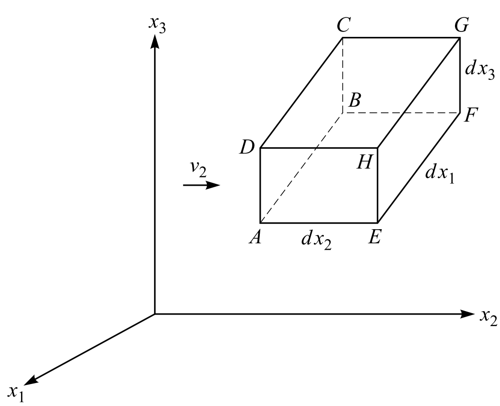


Figure 1.14. Steady flow of a fluid.



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operate on $\mathbf{V}$, and then we keep $\mathbf{V}$ fixed and let $\nabla$ operate on $f$ ($\nabla \cdot f$ is nonsense), and as $\nabla f$ and $\mathbf{V}$ are vectors we complete their multiplication by taking their dot product.

A vector $\mathbf{V}$ is said to be solenoidal if its divergence is zero: $\nabla \cdot \mathbf{V} = 0$.

### *The operator $\nabla^2$, the Laplacian*

The divergence of a vector field is defined by the scalar product of the operator $\nabla$ with the vector field. What is the scalar product of $\nabla$ with itself?

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3\right) \cdot \left(\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3\right)$$
$$= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

This important quantity

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{1.49}$$

is a scalar differential operator which is called the Laplacian, after a French mathematician of the eighteenth century named Laplace. Now, what is the divergence of a gradient?

Since the Laplacian is a scalar differential operator, it does not change the vector character of the field on which it operates. Thus $\nabla^2\phi(\mathbf{r})$ is a scalar field if $\phi(\mathbf{r})$ is a scalar field, and $\nabla^2[\nabla\phi(\mathbf{r})]$ is a vector field because the gradient $\nabla\phi(\mathbf{r})$ is a vector field.

The equation $\nabla^2\phi = 0$ is called Laplace's equation.

### *The curl of a vector*

If $\mathbf{V}(x_1, x_2, x_3)$ is a differentiable vector field, then the curl or rotation of $\mathbf{V}$, written $\nabla \times \mathbf{V}$ (or curl $\mathbf{V}$ or rot $\mathbf{V}$), is defined by the vector product

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix}$$
$$= \hat{e}_1\left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) + \hat{e}_2\left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) + \hat{e}_3\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)$$
$$= \sum_{i,j,k} \varepsilon_{ijk}\hat{e}_i \frac{\partial V_k}{\partial x_j}. \tag{1.50}$$

The result is a vector field. In the expansion of the determinant the operators $\partial/\partial x_i$ must precede $V_i$; $\sum_{ijk}$ stands for $\sum_i \sum_j \sum_k$; and $\varepsilon_{ijk}$ are the permutation symbols: an even permutation of $ijk$ will not change the value of the resulting permutation symbol, but an odd permutation gives an opposite sign. That is,

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}, \quad \text{and}$$
$$\varepsilon_{ijk} = 0 \text{ if two or more indices are equal.}$$

A vector **V** is said to be irrotational if its curl is zero: $\nabla \times \mathbf{V}(\mathbf{r}) = 0$. From this definition we see that the gradient of any scalar field $\phi(\mathbf{r})$ is irrotational. The proof is simple:

$$\nabla \times (\nabla \phi) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \end{vmatrix} \phi(x_1, x_2, x_3) = 0 \tag{1.51}$$

because there are two identical rows in the determinant. Or, in terms of the permutation symbols, we can write $\nabla \times (\nabla \phi)$ as

$$\nabla \times (\nabla \phi) = \sum_{ijk} \varepsilon_{ijk} \hat{e}_i \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi(x_1, x_2, x_3).$$

Now $\varepsilon_{ijk}$ is antisymmetric in $j$, $k$, but $\partial^2/\partial x_j \partial x_k$ is symmetric, hence each term in the sum is always cancelled by another term:

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} + \varepsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} = 0,$$

and consequently $\nabla \times (\nabla \phi) = 0$. Thus, for a conservative vector field **F**, we have curl $\mathbf{F} =$ curl (grad $\phi) = 0$.

We learned above that a vector **V** is solenoidal (or divergence-free) if its divergence is zero. From this we see that the curl of any vector field **V**(**r**) must be solenoidal:

$$\nabla \cdot (\nabla \times \mathbf{V}) = \sum_i \frac{\partial}{\partial x_i} (\nabla \times \mathbf{V})_i = \sum_i \frac{\partial}{\partial x_i} \left( \sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} V_k \right) = 0, \tag{1.52}$$

because $\varepsilon_{ijk}$ is antisymmetric in $i$, $j$.

If $\phi(\mathbf{r})$ is a scalar field and $\mathbf{V}(\mathbf{r})$ is a vector field, then

$$\nabla \times (\phi \mathbf{V}) = \phi (\nabla \times \mathbf{V}) + (\nabla \phi) \times \mathbf{V}. \tag{1.53}$$

We first write

$$\nabla \times (\phi \mathbf{V}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \phi V_1 & \phi V_2 & \phi V_3 \end{vmatrix},$$

then notice that

$$\frac{\partial}{\partial x_1}(\phi V_2) = \phi \frac{\partial V_2}{\partial x_1} + \frac{\partial \phi}{\partial x_1} V_2,$$

so we can expand the determinant in the above equation as a sum of two determinants:

$$\begin{aligned} \nabla \times (\phi \mathbf{V}) &= \phi \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix} + \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial \phi}{\partial x_1} & \dfrac{\partial \phi}{\partial x_2} & \dfrac{\partial \phi}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix} \\ &= \phi(\nabla \times \mathbf{V}) + (\nabla \phi) \times \mathbf{V}. \end{aligned}$$

Alternatively, we can simplify the proof with the help of the permutation symbols $\varepsilon_{ijk}$:

$$\begin{aligned} \nabla \times (\phi \mathbf{V}) &= \sum_{i,j,k} \varepsilon_{ijk} \hat{e}_i \frac{\partial}{\partial x_j}(\phi V_k) \\ &= \phi \sum_{i,j,k} \varepsilon_{ijk} \hat{e}_i \frac{\partial V_k}{\partial x_j} + \sum_{i,j,k} \varepsilon_{ijk} \hat{e}_i \frac{\partial \phi}{\partial x_j} V_k \\ &= \phi(\nabla \times \mathbf{V}) + (\nabla \phi) \times \mathbf{V}. \end{aligned}$$

A vector field that has non-vanishing curl is called a vortex field, and the curl of the field vector is a measure of the vorticity of the vector field.

The physical significance of the curl of a vector is not quite as transparent as that of the divergence. The following example from fluid flow will help us to develop a better feeling. Fig. 1.15 shows that as the component $v_2$ of the velocity $\mathbf{v}$ of the fluid increases with $x_3$, the fluid curls about the $x_1$-axis in a negative sense (rule of the right-hand screw), where $\partial v_2/\partial x_3$ is considered positive. Similarly, a positive curling about the $x_1$-axis would result from $v_3$ if $\partial v_3/\partial x_2$ were positive. Therefore, the total $x_1$ component of the curl of $\mathbf{v}$ is

$$[\text{curl } \mathbf{v}]_1 = \partial v_3/(\partial x_2 - \partial v_2/\partial x_3,$$

which is the same as the $x_1$ component of Eq. (1.50).



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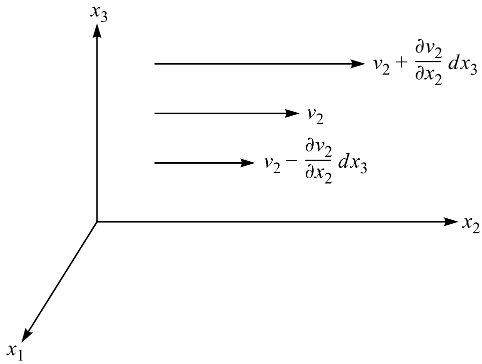


Figure 1.15. Curl of a fluid flow.

## Formulas involving ∇

We now list some important formulas involving the vector differential operator ∇, some of which are recapitulation. In these formulas, **A** and **B** are differentiable vector field functions, and $f$ and $g$ are differentiable scalar field functions of position $(x_1, x_2, x_3)$:

(1) $\nabla(fg) = f\nabla g + g\nabla f$;
(2) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \nabla f \cdot \mathbf{A}$;
(3) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$;
(4) $\nabla \times (\nabla f) = 0$;
(5) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$;
(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \times \mathbf{A}$;
(7) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B}$;
(8) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$;
(9) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$;
(10) $(\mathbf{A} \cdot \nabla)\mathbf{r} = \mathbf{A}$;
(11) $\nabla \cdot \mathbf{r} = 3$;
(12) $\nabla \times \mathbf{r} = 0$;
(13) $\nabla \cdot (r^{-3}\mathbf{r}) = 0$;
(14) $d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} dt$ (**F** a differentiable vector field quantity);
(15) $d\varphi = d\mathbf{r} \cdot \nabla\varphi + \frac{\partial \varphi}{\partial t} dt$ ($\varphi$ a differentiable scalar field quantity).

## Orthogonal curvilinear coordinates

Up to this point all calculations have been performed in rectangular Cartesian coordinates. Many calculations in physics can be greatly simplified by using, instead of the familiar rectangular Cartesian coordinate system, another kind of system which takes advantage of the relations of symmetry involved in the particular problem under consideration. For example, if we are dealing with sphere, we will find it expedient to describe the position of a point in sphere by the spherical coordinates $(r, \theta, \phi)$. Spherical coordinates are a special case of the orthogonal curvilinear coordinate system. Let us now proceed to discuss these more general coordinate systems in order to obtain expressions for the gradient, divergence, curl, and Laplacian. Let the new coordinates $u_1, u_2, u_3$ be defined by specifying the Cartesian coordinates $(x_1, x_2, x_3)$ as functions of $(u_1, u_2, u_3)$:

$$x_1 = f(u_1, u_2, u_3), \quad x_2 = g(u_1, u_2, u_3), \quad x_3 = h(u_1, u_2, u_3), \tag{1.54}$$

where $f$, $g$, $h$ are assumed to be continuous, differentiable. A point $P$ (Fig. 1.16) in space can then be defined not only by the rectangular coordinates $(x_1, x_2, x_3)$ but also by curvilinear coordinates $(u_1, u_2, u_3)$.

If $u_2$ and $u_3$ are constant as $u_1$ varies, $P$ (or its position vector $\mathbf{r}$) describes a curve which we call the $u_1$ coordinate curve. Similarly, we can define the $u_2$ and $u_3$ coordinate curves through $P$. We adopt the convention that the new coordinate system is a right handed system, like the old one. In the new system $d\mathbf{r}$ takes the form:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3.$$

The vector $\partial \mathbf{r}/\partial u_1$ is tangent to the $u_1$ coordinate curve at $P$. If $\hat{u}_1$is a unit vector at $P$ in this direction, then $\hat{u}_1 = \partial \mathbf{r}/\partial u_1 / |\partial \mathbf{r}/\partial u_1|$, so we can write $\partial \mathbf{r}/\partial u_1 = h_1 \hat{u}_1$, where $h_1 = |\partial \mathbf{r}/\partial u_1|$. Similarly we can write $\partial \mathbf{r}/\partial u_2 = h_2 \hat{u}_2$ and $\partial \mathbf{r}/\partial u_3 = h_3 \hat{u}_3$, where $h_2 = |\partial \mathbf{r}/\partial u_2|$ and $h_3 = |\partial \mathbf{r}/\partial u_3|$, respectively. Then $d\mathbf{r}$ can be written

$$d\mathbf{r} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3. \tag{1.55}$$

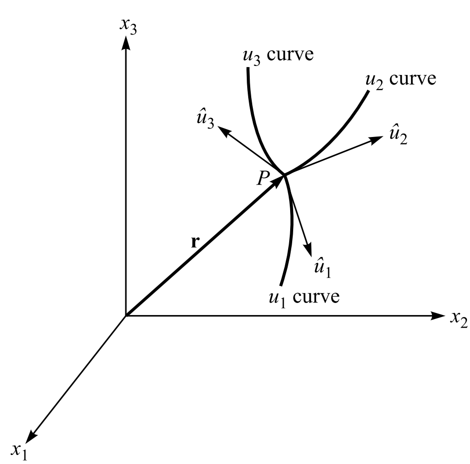


Figure 1.16. Curvilinear coordinates.

The quantities $h_1, h_2, h_3$ are sometimes called scale factors. The unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ are in the direction of increasing $u_1, u_2, u_3$, respectively.

If $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ are mutually perpendicular at any point $P$, the curvilinear coordinates are called orthogonal. In such a case the element of arc length $ds$ is given by

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2. \tag{1.56}$$

Along a $u_1$ curve, $u_2$ and $u_3$ are constants so that $d\mathbf{r} = h_1 du_1 \hat{u}_1$. Then the differential of arc length $ds_1$ along $u_1$ at $P$ is $h_1 du_1$. Similarly the differential arc lengths along $u_2$ and $u_3$ at $P$ are $ds_2 = h_2 du_2$, $ds_3 = h_3 du_3$ respectively.

The volume of the parallelepiped is given by

$$dV = |(h_1 du_1 \hat{u}_1) \cdot (h_2 du_2 \hat{u}_2) \times (h_3 du_3 \hat{u}_3)| = h_1 h_2 h_3 du_1 du_2 du_3$$

since $|\hat{u}_1 \cdot \hat{u}_2 \times \hat{u}_3| = 1$. Alternatively $dV$ can be written as

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1 du_2 du_3 = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3, \tag{1.57}$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} & \dfrac{\partial x_1}{\partial u_3} \\ \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_2}{\partial u_2} & \dfrac{\partial x_2}{\partial u_3} \\ \dfrac{\partial x_3}{\partial u_1} & \dfrac{\partial x_3}{\partial u_2} & \dfrac{\partial x_3}{\partial u_3} \end{vmatrix}$$

is called the Jacobian of the transformation.

We assume that the Jacobian $J \neq 0$ so that the transformation (1.54) is one to one in the neighborhood of a point.

We are now ready to express the gradient, divergence, and curl in terms of $u_1, u_2$, and $u_3$. If $\phi$ is a scalar function of $u_1, u_2$, and $u_3$, then the gradient takes the form

$$\nabla \phi = \operatorname{grad} \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \hat{u}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \hat{u}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \hat{u}_3. \tag{1.58}$$

To derive this, let

$$\nabla \phi = f_1 \hat{u}_1 + f_2 \hat{u}_2 + f_3 \hat{u}_3, \tag{1.59}$$

where $f_1, f_2, f_3$ are to be determined. Since

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \\ &= h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3, \end{aligned}$$

we have

$$d\phi = \nabla\phi \cdot d\mathbf{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3.$$

But

$$d\phi = \frac{\partial \phi}{\partial u_1} du_1 + \frac{\partial \phi}{\partial u_2} du_2 + \frac{\partial \phi}{\partial u_3} du_3,$$

and on equating the two equations, we find

$$f_i = \frac{1}{h_i}\frac{\partial \phi}{\partial u_i}, \qquad i = 1, 2, 3.$$

Substituting these into Eq. (1.57), we obtain the result Eq. (1.58).

From Eq. (1.58) we see that the operator $\nabla$ takes the form

$$\nabla = \frac{\hat{u}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial}{\partial u_3}. \tag{1.60}$$

Because we will need them later, we now proceed to prove the following two relations:

(a) $|\nabla u_i| = h_i^{-1}, i = 1,\ 2,\ 3.$
(b) $\hat{u}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3$ with similar equations for $\hat{u}_2$ and $\hat{u}_3$. (1.61)

Proof: (*a*) Let $\phi = u_1$ in Eq. (1.51), we then obtain $\nabla u_1 = \hat{u}_1/h_1$ and so

$$|\nabla u_1| = |\hat{u}_1| h_1^{-1} = h_1^{-1}, \text{since} \quad |\hat{u}_1| = 1.$$

Similarly by letting $\phi = u_2$ and $u_3$, we obtain the relations for $i = 2$ and 3.

(*b*) From (*a*) we have

$$\nabla u_1 = \hat{u}_1/h_1, \quad \nabla u_2 = \hat{u}_2/h_2, \quad \text{and} \quad \nabla u_3 = \hat{u}_3/h_3.$$

Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{u}_2 \times \hat{u}_3}{h_2 h_3} = \frac{\hat{u}_1}{h_2 h_3} \quad \text{and} \quad \hat{u}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3.$$

Similarly

$$\hat{u}_2 = h_3 h_1 \nabla u_3 \times \nabla u_1 \quad \text{and} \quad \hat{u}_3 = h_1 h_2 \nabla u_1 \times \nabla u_2.$$

We are now ready to express the divergence in terms of curvilinear coordinates. If $\mathbf{A} = A_1\hat{u}_1 + A_2\hat{u}_2 + A_3\hat{u}_3$ is a vector function of orthogonal curvilinear coordinates $u_1$, $u_2$, and $u_3$, the divergence will take the form

$$\nabla \cdot \mathbf{A} = \operatorname{div} \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]. \tag{1.62}$$

To derive (1.62), we first write $\nabla \cdot \mathbf{A}$ as

$$\nabla \cdot \mathbf{A} = \nabla \cdot (A_1 \hat{u}_1) + \nabla \cdot (A_2 \hat{u}_2) + \nabla \cdot (A_3 \hat{u}_3), \tag{1.63}$$



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then, because $\hat{u}_1 = h_1 h_2 \nabla u_2 \times \nabla u_3$, we express $\nabla \cdot (A_1 \hat{u}_1)$ as

$$\nabla \cdot (A_1 \hat{u}_1) = \nabla \cdot (A_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \quad (\hat{u}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3)$$
$$= \nabla (A_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + A_1 h_2 h_3 \nabla \cdot (\nabla u_2 \times \nabla u_3),$$

where in the last step we have used the vector identity: $\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi (\nabla \times \mathbf{A})$. Now $\nabla u_i = \hat{u}_i / h_i, i = 1, 2, 3$, so $\nabla \cdot (A_1 \hat{u}_1)$ can be rewritten as

$$\nabla \cdot (A_1 \hat{u}_1) = \nabla (A_1 h_2 h_3) \cdot \frac{\hat{u}_2}{h_2} \times \frac{\hat{u}_3}{h_3} + 0 = \nabla (A_1 h_2 h_3) \cdot \frac{\hat{u}_1}{h_2 h_3}.$$

The gradient $\nabla (A_1 h_2 h_3)$ is given by Eq. (1.58), and we have

$$\nabla \cdot (A_1 \hat{u}_1) = \left[ \frac{\hat{u}_1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\hat{u}_2}{h_2} \frac{\partial}{\partial u_2} (A_1 h_2 h_3) + \frac{\hat{u}_3}{h_3} \frac{\partial}{\partial u_3} (A_1 h_2 h_3) \right] \cdot \frac{\hat{u}_1}{h_2 h_3}$$
$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3).$$

Similarly, we have

$$\nabla \cdot (A_2 \hat{u}_2) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (A_2 h_3 h_1), \quad \text{and} \quad \nabla \cdot (A_3 \hat{u}_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (A_3 h_2 h_1).$$

Substituting these into Eq. (1.63), we obtain the result, Eq. (1.62).

In the same manner we can derive a formula for curl **A**. We first write it as

$$\nabla \times \mathbf{A} = \nabla \times (A_1 \hat{u}_1 + A_2 \hat{u}_2 + A_3 \hat{u}_3)$$

and then evaluate $\nabla \times A_i \hat{u}_i$.

Now $\hat{u}_i = h_i \nabla u_i, i = 1, 2, 3$, and we express $\nabla \times (A_1 \hat{u}_1)$ as

$$\nabla \times (A_1 \hat{u}_1) = \nabla \times (A_1 h_1 \nabla u_1)$$
$$= \nabla (A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1$$
$$= \nabla (A_1 h_1) \times \frac{\hat{u}_1}{h_1} + 0$$
$$= \left[ \frac{\hat{u}_1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_1) + \frac{\hat{u}_2}{h_2} \frac{\partial}{\partial u_2} (A_2 h_2) + \frac{\hat{u}_3}{h_3} \frac{\partial}{\partial u_3} (A_3 h_3) \right] \times \frac{\hat{u}_1}{h_1}$$
$$= \frac{\hat{u}_2}{h_3 h_1} \frac{\partial}{\partial u_3} (A_1 h_1) - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial}{\partial u_2} (A_1 h_1),$$

with similar expressions for $\nabla \times (A_2 \hat{u}_2)$ and $\nabla \times (A_3 \hat{u}_3)$. Adding these together, we get $\nabla \times \mathbf{A}$ in orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{A} = \frac{\hat{u}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right] + \frac{\hat{u}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\partial}{\partial u_1}(A_3 h_3)\right]$$
$$+ \frac{\hat{u}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{\partial}{\partial u_2}(A_1 h_1)\right]. \tag{1.64}$$

This can be written in determinant form:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}. \tag{1.65}$$

We now express the Laplacian in orthogonal curvilinear coordinates. From Eqs. (1.58) and (1.62) we have

$$\nabla \phi = \text{grad}\,\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\hat{u} + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\hat{u}_3,$$

$$\nabla \cdot \mathbf{A} = \text{div}\,\mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right].$$

If $\mathbf{A} = \nabla \phi$, then $A_i = (1/h_i)\partial \phi / \partial u_i$, $i = 1, 2, 3$; and

$$\nabla \cdot \mathbf{A} = \nabla \cdot \nabla \phi = \nabla^2 \phi$$
$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right]. \tag{1.66}$$

## Special orthogonal coordinate systems

There are at least nine special orthogonal coordinates systems, the most common and useful ones are the cylindrical and spherical coordinates; we introduce these two coordinates in this section.

### *Cylindrical coordinates* $(\rho, \phi, z)$

$$u_1 = \rho, u_2 = \phi, u_3 = z; \quad \text{and} \quad \hat{u}_1 = e_\rho, \hat{u}_2 = e_\phi \hat{u}_3 = e_z.$$

From Fig. 1.17 we see that

$$x_1 = \rho \cos \phi, x_2 = \rho \sin \phi, x_3 = z$$

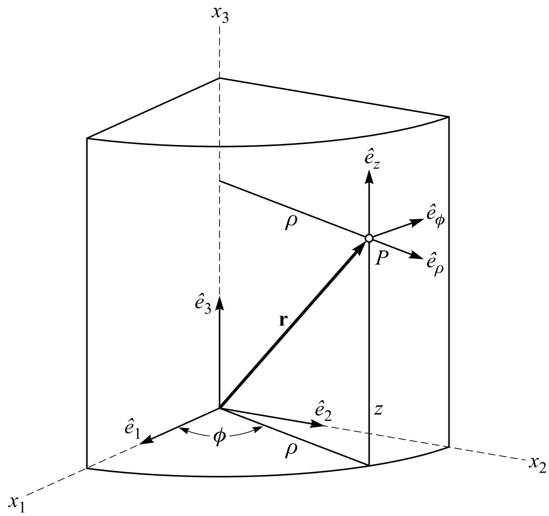


Figure 1.17. Cylindrical coordinates.

where

$$\rho \geq 0, 0 \leq \phi \leq 2\pi, -\infty < z < \infty.$$

The square of the element of arc length is given by

$$ds^2 = h_1^2(d\rho)^2 + h_2^2(d\phi)^2 + h_3^2(dz)^2.$$

To find the scale factors $h_i$, we notice that $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$ where

$$\mathbf{r} = \rho \cos \phi e_1 + \rho \sin \phi e_2 + z e_3.$$

Thus

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.$$

Equating the two $ds^2$, we find the scale factors:

$$h_1 = h_\rho = 1, h_2 = h_\phi = \rho, h_3 = h_z = 1. \tag{1.67}$$

From Eqs. (1.58), (1.62), (1.64), and (1.66) we find the gradient, divergence, curl, and Laplacian in cylindrical coordinates:

$$\nabla \Phi = \frac{\partial \Phi}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi} e_\phi + \frac{\partial \Phi}{\partial z} e_z, \tag{1.68}$$

where $\Phi = \Phi(\rho, \phi, z)$ is a scalar function;

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{\partial A_\phi}{\partial \phi} + \frac{\partial}{\partial z} (\rho A_z) \right]; \tag{1.69}$$

where

$$\mathbf{A} = A_\rho e_\rho + A_\phi e_\phi + A_z e_z;$$

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\phi & e_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}; \tag{1.70}$$

and

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}. \tag{1.71}$$

### ***Spherical coordinates*** $(r, \theta, \phi)$

$$u_1 = r, u_2 = \theta, u_3 = \phi; \hat{u}_1 = e_r, \hat{u}_2 = e_\theta, \hat{u}_3 = e_\phi$$

From Fig. 1.18 we see that

$$x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi, x_3 = r\cos\theta.$$

Now

$$ds^2 = h_1^2(dr)^2 + h_2^2(d\theta)^2 + h_3^2(d\phi)^2$$

but

$$\mathbf{r} = r\sin\theta\cos\phi\hat{e}_1 + r\sin\theta\sin\phi\hat{e}_2 + r\cos\theta\hat{e}_3,$$

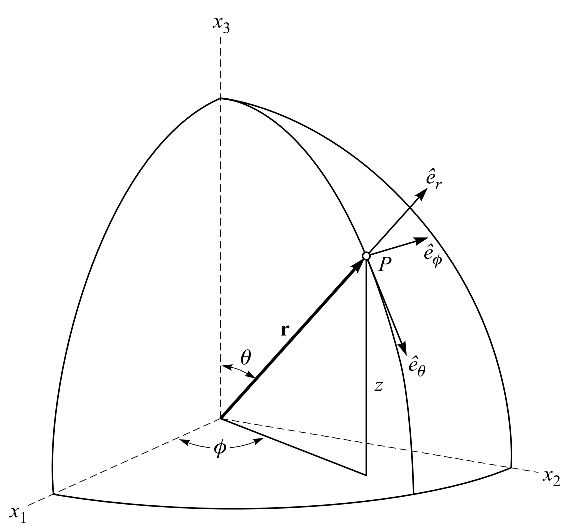


Figure 1.18. Spherical coordinates.



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so

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2.$$

Equating the two $ds^2$, we find the scale factors: $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$. We then find, from Eqs. (1.58), (1.62), (1.64), and (1.66), the gradient, divergence, curl, and the Laplacian in spherical coordinates:

$$\nabla\Phi = \hat{e}_r \frac{\partial\Phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}; \tag{1.72}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2 A_r) + r\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + r\frac{\partial A_\phi}{\partial\phi}\right]; \tag{1.73}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_r & r\sin\theta A_\phi \end{vmatrix}; \tag{1.74}$$

$$\nabla^2\Phi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right]. \tag{1.75}$$

## Vector integration and integral theorems

Having discussed vector differentiation, we now turn to a discussion of vector integration. After defining the concepts of line, surface, and volume integrals of vector fields, we then proceed to the important integral theorems of Gauss, Stokes, and Green.

The integration of a vector, which is a function of a single scalar $u$, can proceed as ordinary scalar integration. Given a vector

$$\mathbf{A}(u) = A_1(u)\hat{e}_1 + A_2(u)\hat{e}_2 + A_3(u)\hat{e}_3,$$

then

$$\int \mathbf{A}(u)du = \hat{e}_1\int A_1(u)du + \hat{e}_2\int A_2(u)du + \hat{e}_3\int A_3(u)du + \mathbf{B},$$

where $\mathbf{B}$ is a constant of integration, a constant vector. Now consider the integral of the scalar product of a vector $\mathbf{A}(x_1, x_2, x_3)$ and $d\mathbf{r}$ between the limit $P_1(x_1, x_2, x_3)$ and $P_2(x_1, x_2, x_3)$:



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$$\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} = \int_{P_1}^{P_2} (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) \cdot (dx_1\hat{e}_1 + dx_2\hat{e}_2 + dx_3\hat{e}_3)$$
$$= \int_{P_1}^{P_2} A_1(x_1, x_2, x_3)dx_1 + \int_{P_1}^{P_2} A_2(x_1, x_2, x_3)dx_2$$
$$+ \int_{P_1}^{P_2} A_3(x_1, x_2, x_3)dx_3.$$

Each integral on the right hand side requires for its execution more than a knowledge of the limits. In fact, the three integrals on the right hand side are not completely defined because in the first integral, for example, we do not the know value of $x_2$ and $x_3$ in $A_1$:

$$I_1 = \int_{P_1}^{P_2} A_1(x_1, x_2, x_3)dx_1. \tag{1.76}$$

What is needed is a statement such as

$$x_2 = f(x_1), x_3 = g(x_1) \tag{1.77}$$

that specifies $x_2$, $x_3$ for each value of $x_1$. The integrand now reduces to $A_1(x_1, x_2, x_3) = A_1(x_1, f(x_1), g(x_1)) = B_1(x_1)$ so that the integral $I_1$ becomes well defined. But its value depends on the constraints in Eq. (1.77). The constraints specify paths on the $x_1x_2$ and $x_3x_1$ planes connecting the starting point $P_1$ to the end point $P_2$. The $x_1$ integration in (1.76) is carried out along these paths. It is a path-dependent integral and is called a line integral (or a path integral). It is very helpful to keep in mind that: *when the number of integration variables is less than the number of variables in the integrand, the integral is not yet completely defined and it is path-dependent*. However, if the scalar product $\mathbf{A} \cdot d\mathbf{r}$ is equal to an exact differential, $\mathbf{A} \cdot d\mathbf{r} = d\varphi = \nabla\varphi \cdot d\mathbf{r}$, the integration depends only upon the limits and is therefore path-independent:

$$\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} = \int_{P_1}^{P_2} d\varphi = \varphi_2 - \varphi_1.$$

A vector field $\mathbf{A}$ which has above (path-independent) property is termed conservative. It is clear that the line integral above is zero along any close path, and the curl of a conservative vector field is zero ($\nabla \times \mathbf{A} = \nabla \times (\nabla\varphi) = 0$). A typical example of a conservative vector field in mechanics is a conservative force.

The surface integral of a vector function $\mathbf{A}(x_1, x_2, x_3)$ over the surface $S$ is an important quantity; it is defined to be

$$\int_S \mathbf{A} \cdot d\mathbf{a},$$



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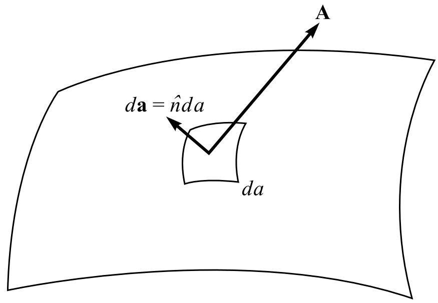


Figure 1.19. Surface integral over a surface $S$.

where the surface integral symbol $\int_s$ stands for a double integral over a certain surface $S$, and $d\mathbf{a}$ is an element of area of the surface (Fig. 1.19), a vector quantity. We attribute to $d\mathbf{a}$ a magnitude $da$ and also a direction corresponding the normal, $\hat{n}$, to the surface at the point in question, thus

$$d\mathbf{a} = \hat{n}da.$$

The normal $\hat{n}$ to a surface may be taken to lie in either of two possible directions. But if $da$ is part of a closed surface, the sign of $\hat{n}$ relative to $da$ is so chosen that it points outward away from the interior. In rectangular coordinates we may write

$$d\mathbf{a} = \hat{e}_1 da_1 + \hat{e}_2 da_2 + \hat{e}_3 da_3 = \hat{e}_1 dx_2 dx_3 + \hat{e}_2 dx_3 dx_1 + \hat{e}_3 dx_1 dx_2.$$

If a surface integral is to be evaluated over a closed surface $S$, the integral is written as

$$\oint_S \mathbf{A} \cdot d\mathbf{a}.$$

Note that this is different from a closed-path line integral. When the path of integration is closed, the line integral is write it as

$$\oint_\Gamma \mathbf{A} \cdot d\mathbf{s},$$

where $\Gamma$ specifies the closed path, and $d\mathbf{s}$ is an element of length along the given path. By convention, $d\mathbf{s}$ is taken positive along the direction in which the path is traversed. Here we are only considering simple closed curves. A simple closed curve does not intersect itself anywhere.

### *Gauss' theorem (the divergence theorem)*

This theorem relates the surface integral of a given vector function and the volume integral of the divergence of that vector. It was introduced by Joseph Louis Lagrange and was first used in the modern sense by George Green. Gauss'



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name is associated with this theorem because of his extensive work on general problems of double and triple integrals.

If a continuous, differentiable vector field **A** is defined in a simply connected region of volume $V$ bounded by a closed surface $S$, then the theorem states that

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{a}, \tag{1.78}$$

where $dV = dx_1 dx_2 dx_3$. A simple connected region $V$ has the property that every simple closed curve within it can be continuously shrunk to a point without leaving the region. To prove this, we first write

$$\int_V \nabla \cdot \mathbf{A} dV = \int_V \sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} dV,$$

then integrate the right hand side with respect to $x_1$ while keeping $x_2 x_3$ constant, thus summing up the contribution from a rod of cross section $dx_2 dx_3$ (Fig. 1.20). The rod intersects the surface $S$ at the points $P$ and $Q$ and thus defines two elements of area $d\mathbf{a}_P$ and $d\mathbf{a}_Q$:

$$\int_V \frac{\partial A_1}{\partial x_1} dV = \oint_S dx_2 dx_3 \int_P^Q \frac{\partial A_1}{\partial x_1} dx_1 = \oint_S dx_2 dx_3 \int_P^Q dA_1,$$

where we have used the relation $dA_1 = (\partial A_1/\partial x_1) dx_1$ along the rod. The last integration on the right hand side can be performed at once and we have

$$\int_V \frac{\partial A_1}{\partial x_1} dV = \oint_S [A_1(Q) - A_1(P)] dx_2 dx_3,$$

where $A_1(Q)$ denotes the value of $A_1$ evaluated at the coordinates of the point $Q$, and similarly for $A_1(P)$.

The component of the surface element $d\mathbf{a}$ which lies in the $x_1$-direction is $da_1 = dx_2 dx_3$ at the point $Q$, and $da_1 = -dx_2 dx_3$ at the point $P$. The minus sign

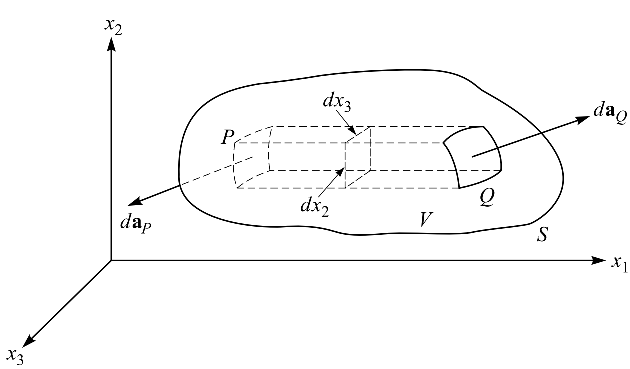


Figure 1.20. A square tube of cross section $dx_2 dx_3$.

arises since the $x_1$ component of $d\mathbf{a}$ at $P$ is in the direction of negative $x_1$. We can now rewrite the above integral as

$$\int_V \frac{\partial A_1}{\partial x_1} dV = \int_{S_Q} A_1(Q) da_1 + \int_{S_P} A_1(P) da_1,$$

where $S_Q$ denotes that portion of the surface for which the $x_1$ component of the outward normal to the surface element $da_1$ is in the positive $x_1$-direction, and $S_P$ denotes that portion of the surface for which $da_1$ is in the negative direction. The two surface integrals then combine to yield the surface integral over the entire surface $S$ (if the surface is sufficiently concave, there may be several such as right hand and left hand portions of the surfaces):

$$\int_V \frac{\partial A_1}{\partial x_1} dV = \oint_S A_1 da_1.$$

Similarly we can evaluate the $x_2$ and $x_3$ components. Summing all these together, we have Gauss' theorem:

$$\int_V \sum_i \frac{\partial A_i}{\partial x_i} dV = \oint_S \sum_i A_i da_i \qquad \text{or} \qquad \int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{a}.$$

We have proved Gauss' theorem for a simply connected region (a volume bounded by a single surface), but we can extend the proof to a multiply connected region (a region bounded by several surfaces, such as a hollow ball). For interested readers, we recommend the book *Electromagnetic Fields*, Roald K. Wangsness, John Wiley, New York, 1986.

### *Continuity equation*

Consider a fluid of density $\rho(\mathbf{r})$ which moves with velocity $\mathbf{v}(\mathbf{r})$ in a certain region. If there are no sources or sinks, the following continuity equation must be satisfied:

$$\partial\rho(\mathbf{r})/\partial t + \nabla \cdot \mathbf{j}(\mathbf{r}) = 0, \tag{1.79}$$

where $\mathbf{j}$ is the current

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \tag{1.79a}$$

and Eq. (1.79) is called the continuity equation for a conserved current.

To derive this important equation, let us consider an arbitrary surface $S$ enclosing a volume $V$ of the fluid. At any time the mass of fluid within $V$ is $M = \int_V \rho dV$ and the time rate of mass increase (due to mass flowing into $V$) is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV,$$



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while the mass of fluid leaving $V$ per unit time is

$$\int_S \rho \mathbf{v} \cdot \hat{n} ds = \int_V \nabla \cdot (\rho \mathbf{v}) dV,$$

where Gauss' theorem is used in changing the surface integral to volume integral. Since there is neither a source nor a sink, mass conservation requires an exact balance between these effects:

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \mathbf{v}) dV, \quad \text{or} \quad \int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0.$$

Also since $V$ is arbitrary, mass conservation requires that the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} \nabla \cdot \mathbf{j} = 0$$

must be satisfied everywhere in the region.

### *Stokes' theorem*

This theorem relates the line integral of a vector function and the surface integral of the curl of that vector. It was first discovered by Lord Kelvin in 1850 and rediscovered by George Gabriel Stokes four years later.

If a continuous, differentiable vector field $\mathbf{A}$ is defined a three-dimensional region $V$, and $S$ is a regular open surface embedded in $V$ bounded by a simple closed curve $\Gamma$, the theorem states that

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_\Gamma \mathbf{A} \cdot d\mathbf{l}; \tag{1.80}$$

where the line integral is to be taken completely around the curve $\Gamma$ and $d\mathbf{l}$ is an element of line (Fig. 1.21).

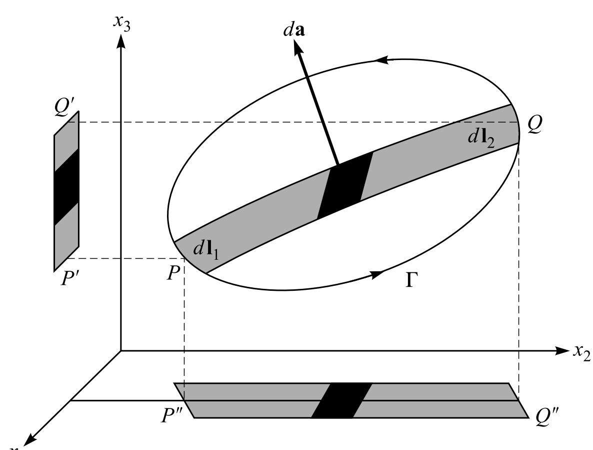


Figure 1.21. Relation between $d\mathbf{a}$ and $d\mathbf{l}$ in defining curl.



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The surface $S$, bounded by a simple closed curve, is an open surface; and the normal to an open surface can point in two opposite directions. We adopt the usual convention, namely the right hand rule: when the fingers of the right hand follow the direction of $d\mathbf{l}$, the thumb points in the $d\mathbf{a}$ direction, as shown in Fig. 1.21.

Note that Eq. (1.80) does not specify the shape of the surface $S$ other than that it be bounded by $\Gamma$; thus there are many possibilities in choosing the surface. But Stokes' theorem enables us to reduce the evaluation of surface integrals which depend upon the shape of the surface to the calculation of a line integral which depends only on the values of $\mathbf{A}$ along the common perimeter.

To prove the theorem, we first expand the left hand side of Eq. (1.80); with the aid of Eq. (1.50), it becomes

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \int_S \left( \frac{\partial A_1}{\partial x_3} da_2 - \frac{\partial A_1}{\partial x_2} da_3 \right) + \int_S \left( \frac{\partial A_2}{\partial x_1} da_3 - \frac{\partial A_2}{\partial x_3} da_1 \right) + \int_S \left( \frac{\partial A_3}{\partial x_2} da_1 - \frac{\partial A_3}{\partial x_1} da_2 \right), \tag{1.81}$$

where we have grouped the terms by components of $\mathbf{A}$. We next subdivide the surface $S$ into a large number of small strips, and integrate the first integral on the right hand side of Eq. (1.81), denoted by $I_1$, over one such a strip of width $dx_1$, which is parallel to the $x_2x_3$ plane and a distance $x_1$ from it, as shown in Fig. 1.21. Then, by integrating over $x_1$, we sum up the contributions from all of the strips.

Fig. 1.21 also shows the projections of the strip on the $x_1x_3$ and $x_1x_2$ planes that will help us to visualize the orientation of the surface. The element area $d\mathbf{a}$ is shown at an intermediate stage of the integration, when the direction angles have values such that $\alpha$ and $\gamma$ are less than $90^\circ$ and $\beta$ is greater than $90^\circ$. Thus, $da_2 = -dx_1dx_3$ and $da_3 = dx_1dx_2$ and we can write

$$I_1 = -\int_{\text{strips}} dx_1 \int_P^Q \left( \frac{\partial A_1}{\partial x_2} dx_2 + \frac{\partial A_1}{\partial x_3} dx_3 \right). \tag{1.82}$$

Note that $dx_2$ and $dx_3$ in the parentheses are not independent because $x_2$ and $x_3$ are related by the equation for the surface $S$ and the value of $x_1$ involved. Since the second integral in Eq. (1.82) is being evaluated on the strip from $P$ to $Q$ for which $x_1 = \text{const.}$, $dx_1 = 0$ and we can add $(\partial A_1/\partial x_1)dx_1 = 0$ to the integrand to make it $dA_1$:

$$\frac{\partial A_1}{\partial x_1} dx_1 + \frac{\partial A_1}{\partial x_2} dx_2 + \frac{\partial A_1}{\partial x_3} dx_3 = dA_1.$$

And Eq. (1.82) becomes

$$I_1 = -\int_{\text{strips}} dx_1 \int_P^Q dA_1 = \int_{\text{strips}} [A_1(P) - A_1(Q)] dx_1.$$

Next we consider the line integral of **A** around the lines bounding each of the small strips. If we trace each one of these lines in the same sense as we trace the path $\Gamma$, then we will trace all of the interior lines twice (once in each direction) and all of the contributions to the line integral from the interior lines will cancel, leaving only the result from the boundary line $\Gamma$. Thus, the sum of all of the line integrals around the small strips will equal the line integral $\Gamma$ of $A_1$:

$$\int_S \left( \frac{\partial A_1}{\partial x_3} da_2 - \frac{\partial A_1}{\partial x_2} da_3 \right) = \oint_\Gamma A_1 dl_1. \tag{1.83}$$

Similarly, the last two integrals of Eq. (1.81) can be shown to have the respective values

$$\oint_\Gamma A_2 dl_2 \quad \text{and} \quad \oint_\Gamma A_3 dl_3.$$

Substituting these results and Eq. (1.83) into Eq. (1.81) we obtain Stokes' theorem:

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_\Gamma (A_1 dl_1 + A_2 dl_2 + A_3 dl_3) = \oint_\Gamma \mathbf{A} \cdot d\mathbf{l}.$$

Stokes' theorem in Eq. (1.80) is valid whether or not the closed curve $\Gamma$ lies in a plane, because in general the surface $S$ is not a planar surface. Stokes' theorem holds for any surface bounded by $\Gamma$.

In fluid dynamics, the curl of the velocity field $\mathbf{v}(\mathbf{r})$ is called its *vorticity* (for example, the whirls that one creates in a cup of coffee on stirring it). If the velocity field is derivable from a potential

$$\mathbf{v}(\mathbf{r}) = -\nabla \phi(\mathbf{r})$$

it must be irrotational (see Eq. (1.51)). For this reason, an irrotational flow is also called a potential flow, which describes a steady flow of the fluid, free of vortices and eddies.

One of Maxwell's equations of electromagnetism (Ampère's law) states that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j},$$

where **B** is the magnetic induction, **j** is the current density (per unit area), and $\mu_0$ is the permeability of free space. From this equation, current densities may be visualized as vortices of **B**. Applying Stokes' theorem, we can rewrite Ampère's law as

$$\oint_\Gamma \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{j} \cdot d\mathbf{a} = \mu_0 I;$$

it states that the circulation of the magnetic induction is proportional to the total current $I$ passing through the surface $S$ enclosed by $\Gamma$.

### *Green's theorem*

Green's theorem is an important corollary of the divergence theorem, and it has many applications in many branches of physics. Recall that the divergence theorem Eq. (1.78) states that

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{a}.$$

Let $\mathbf{A} = \psi \mathbf{B}$, where $\psi$ is a scalar function and $\mathbf{B}$ a vector function, then $\nabla \cdot \mathbf{A}$ becomes

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\psi \mathbf{B}) = \psi \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \psi.$$

Substituting these into the divergence theorem, we have

$$\oint_S \psi \mathbf{B} \cdot d\mathbf{a} = \int_V (\psi \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \psi) dV. \tag{1.84}$$

If $\mathbf{B}$ represents an irrotational vector field, we can express it as a gradient of a scalar function, say, $\varphi$:

$$\mathbf{B} \equiv \nabla \varphi.$$

Then Eq. (1.84) becomes

$$\oint_S \psi \mathbf{B} \cdot d\mathbf{a} = \int_V [\psi \nabla \cdot (\nabla \varphi) + (\nabla \varphi) \cdot (\nabla \psi)] dV. \tag{1.85}$$

Now

$$\mathbf{B} \cdot d\mathbf{a} = (\nabla \varphi) \cdot \hat{n} da.$$

The quantity $(\nabla \varphi) \cdot \hat{n}$ represents the rate of change of $\phi$ in the direction of the outward normal; it is called the normal derivative and is written as

$$(\nabla \varphi) \cdot \hat{n} \equiv \partial \varphi / \partial n.$$

Substituting this and the identity $\nabla \cdot (\nabla \varphi) = \nabla^2 \varphi$ into Eq. (1.85), we have

$$\oint_S \psi \frac{\partial \varphi}{\partial n} da = \int_V [\psi \nabla^2 \varphi + \nabla \varphi \cdot \nabla \psi] dV. \tag{1.86}$$

Eq. (1.86) is known as Green's theorem in the first form.

Now let us interchange $\varphi$ and $\psi$, then Eq. (1.86) becomes

$$\oint_S \varphi \frac{\partial \psi}{\partial n} da = \int_V [\varphi \nabla^2 \psi + \nabla \varphi \cdot \nabla \psi] dV.$$

Subtracting this from Eq. (1.85):

$$\oint_S \left( \psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) da = \int_V \left( \psi \nabla^2 \varphi - \varphi \nabla^2 \psi \right) dV. \tag{1.87}$$



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This important result is known as the second form of Green's theorem, and has many applications.

### *Green's theorem in the plane*

Consider the two-dimensional vector field $\mathbf{A} = M(x_1, x_2)\hat{e}_1 + N(x_1, x_2)\hat{e}_2$. From Stokes' theorem

$$\oint_\Gamma \mathbf{A} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \int_S \left( \frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2} \right) dx_1 dx_2, \tag{1.88}$$

which is often called Green's theorem in the plane.

Since $\oint_\Gamma \mathbf{A} \cdot d\mathbf{r} = \oint_\Gamma (M dx_1 + N dx_2)$, Green's theorem in the plane can be written as

$$\oint_\Gamma M dx_1 + N dx_2 = \int_S \left( \frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2} \right) dx_1 dx_2. \tag{1.88a}$$

As an illustrative example, let us apply Green's theorem in the plane to show that the area bounded by a simple closed curve $\Gamma$ is given by

$$\frac{1}{2} \oint_\Gamma x_1 dx_2 - x_2 dx_1.$$

Into Green's theorem in the plane, let us put $M = -x_2, N = x_1$, giving

$$\oint_\Gamma x_1 dx_2 - x_2 dx_1 = \int_S \left( \frac{\partial}{\partial x_1} x_1 - \frac{\partial}{\partial x_2}(-x_2) \right) dx_1 dx_2 = 2 \int_S dx_1 dx_2 = 2A,$$

where $A$ is the required area. Thus $A = \frac{1}{2} \oint_\Gamma x_1 dx_2 - x_2 dx_1$.

### Helmholtz's theorem

The divergence and curl of a vector field play very important roles in physics. We learned in previous sections that a divergence-free field is solenoidal and a curl-free field is irrotational. We may classify vector fields in accordance with their being solenoidal and/or irrotational. A vector field $\mathbf{V}$ is:

(1) Solenoidal and irrotational if $\nabla \cdot \mathbf{V} = 0$ and $\nabla \times \mathbf{V} = 0$. A static electric field in a charge-free region is a good example.
(2) Solenoidal if $\nabla \cdot \mathbf{V} = 0$ but $\nabla \times \mathbf{V} \neq 0$. A steady magnetic field in a current-carrying conductor meets these conditions.
(3) Irrotational if $\nabla \times \mathbf{V} = 0$ but $\nabla \cdot \mathbf{V} = 0$. A static electric field in a charged region is an irrotational field.

The most general vector field, such as an electric field in a charged medium with a time-varying magnetic field, is neither solenoidal nor irrotational, but can be



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considered as the sum of a solenoidal field and an irrotational field. This is made clear by Helmholtz's theorem, which can be stated as (C. W. Wong: *Introduction to Mathematical Physics*, Oxford University Press, Oxford 1991; p. 53):

> A vector field is uniquely determined by its divergence and curl in a region of space, and its normal component over the boundary of the region. In particular, if both divergence and curl are specified everywhere and if they both disappear at infinity sufficiently rapidly, then the vector field can be written as a unique sum of an irrotational part and a solenoidal part.

In other words, we may write

$$\mathbf{V}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}), \tag{1.89}$$

where $-\nabla\phi$ is the irrotational part and $\nabla \times \mathbf{A}$ is the solenoidal part, and $\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are called the scalar and the vector potential, respectively, of $\mathbf{V}(\mathbf{r})$. If both $\mathbf{A}$ and $\phi$ can be determined, the theorem is verified. How, then, can we determine $\mathbf{A}$ and $\phi$? If the vector field $\mathbf{V}(\mathbf{r})$ is such that

$$\nabla \cdot \mathbf{V}(\mathbf{r}) = \rho, \quad \text{and} \quad \nabla \times \mathbf{V}(\mathbf{r}) = \mathbf{v},$$

then we have

$$\nabla \cdot \mathbf{V}(\mathbf{r}) = \rho = -\nabla \cdot (\nabla\phi) + \nabla \cdot (\nabla \times \mathbf{A})$$

or

$$\nabla^2\phi = -\rho,$$

which is known as Poisson's equation. Next, we have

$$\nabla \times \mathbf{V}(\mathbf{r}) = \mathbf{v} = \nabla \times [-\nabla\phi + \nabla \times \mathbf{A}(\mathbf{r})]$$

or

$$\nabla^2\mathbf{A} = \mathbf{v};$$

or in component, we have

$$\nabla^2 A_i = v_i, i = 1, 2, 3$$

where these are also Poisson's equations. Thus, both $\mathbf{A}$ and $\phi$ can be determined by solving Poisson's equations.

## Some useful integral relations

These relations are closely related to the general integral theorems that we have proved in preceding sections.

(1) The line integral along a curve $C$ between two points $a$ and $b$ is given by

$$\int_a^b (\nabla\phi) \cdot d\mathbf{l} = \phi(b) - \phi(a). \tag{1.90}$$



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Proof:

$$\int_a^b (\nabla\phi)\cdot d\mathbf{l} = \int_a^b \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \left(\frac{d\phi}{dt}\right)dt = \phi(b) - \phi(a).$$

(2)
$$\oint_S \frac{\partial\varphi}{\partial n}da = \int_V \nabla^2\varphi dV. \tag{1.91}$$

Proof: Set $\psi = 1$ in Eq. (1.87), then $\partial\psi/\partial n = 0 = \nabla^2\psi$ and Eq. (1.87) reduces to Eq. (1.91).

(3)
$$\int_V \nabla\varphi dV = \oint_S \varphi\hat{n}da. \tag{1.92}$$

Proof: In Gauss' theorem (1.78), let $\mathbf{A} = \varphi\mathbf{C}$, where $\mathbf{C}$ is constant vector. Then we have

$$\int_V \nabla\cdot(\varphi\mathbf{C})dV = \int_S \varphi\mathbf{C}\cdot\hat{n}da.$$

Since

$$\nabla\cdot(\varphi\mathbf{C}) = \nabla\varphi\cdot\mathbf{C} = \mathbf{C}\cdot\nabla\varphi \quad \text{and} \quad \varphi\mathbf{C}\cdot\hat{n} = \mathbf{C}\cdot(\varphi\hat{n}),$$

we have

$$\int_V \mathbf{C}\cdot\nabla\varphi dV = \int_S \mathbf{C}\cdot(\varphi\hat{n})da.$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C}\cdot\int_V \nabla\varphi dV = \mathbf{C}\cdot\int_S (\varphi\hat{n})da$$

and since $\mathbf{C}$ is an arbitrary constant vector, we have

$$\int_V \nabla\varphi dV = \oint_S \varphi\hat{n}da.$$

(4)
$$\int_V \nabla\times\mathbf{B}dV = \int_S \hat{n}\times\mathbf{B}da \tag{1.93}$$



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Proof: In Gauss' theorem (1.78), let $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is a constant vector. We then have

$$\int_V \nabla \cdot (\mathbf{B} \times \mathbf{C}) dV = \int_S (\mathbf{B} \times \mathbf{C}) \cdot \hat{n} da.$$

Since $\nabla \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{B})$ and $(\mathbf{B} \times \mathbf{C}) \cdot \hat{n} = \mathbf{B} \cdot (\mathbf{C} \times \hat{n}) = (\mathbf{C} \times \hat{n}) \cdot \mathbf{B} = \mathbf{C} \cdot (\hat{n} \times \mathbf{B})$,

$$\int_V \mathbf{C} \cdot (\nabla \times \mathbf{B}) dV = \int_S \mathbf{C} \cdot (\hat{n} \times \mathbf{B}) da.$$

Taking $\mathbf{C}$ outside the integrals

$$\mathbf{C} \cdot \int_V (\nabla \times \mathbf{B}) dV = \mathbf{C} \cdot \int_S (\hat{n} \times \mathbf{B}) da$$

and since $\mathbf{C}$ is an arbitrary constant vector, we have

$$\int_V \nabla \times \mathbf{B} dV = \int_S \hat{n} \times \mathbf{B} da.$$

## Tensor analysis

Tensors are a natural generalization of vectors. The beginnings of tensor analysis can be traced back more than a century to Gauss' works on curved surfaces. Today tensor analysis finds applications in theoretical physics (for example, general theory of relativity, mechanics, and electromagnetic theory) and to certain areas of engineering (for example, aerodynamics and fluid mechanics). The general theory of relativity uses tensor calculus of curved space-time, and engineers mainly use tensor calculus of Euclidean space. Only general tensors are considered in this section. The general definition of a tensor is given, followed by a concise discussion of tensor algebra and tensor calculus (covariant differentiation).

Tensors are defined by means of their properties of transformation under coordinate transformation. Let us consider the transformation from one coordinate system $(x^1, x^2, \ldots, x^N)$ to another $(x'^1, x'^2, \ldots, x'^N)$ in an $N$-dimensional space $V_N$. Note that in writing $x^\mu$, the index $\mu$ is a superscript and should not be mistaken for an exponent. In three-dimensional space we use subscripts. We now use superscripts in order that we may maintain a 'balancing' of the indices in all the general equations. The meaning of 'balancing' will become clear a little later. When we transform the coordinates, their differentials transform according to the relation

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu. \tag{1.94}$$



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Here we have used Einstein's summation convention: repeated indexes which appear once in the lower and once in the upper position are automatically summed over. Thus,

$$\sum_{\mu=1}^{N} A_\mu A^\mu = A_\mu A^\mu.$$

It is important to remember that indexes repeated in the lower part or upper part alone are not summed over. An index which is repeated and over which summation is implied is called a dummy index. Clearly, a dummy index can be replaced by any other index that does not appear in the same term.

## Contravariant and covariant vectors

A set of $N$ quantities $A^\mu(\mu = 1, 2, \ldots, N)$ which, under a coordinate change, transform like the coordinate differentials, are called the components of a contravariant vector or a contravariant tensor of the first rank or first order:

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu. \tag{1.95}$$

This relation can easily be inverted to express $A'^\nu$ in terms of $A^\mu$. We shall leave this as homework for the reader (Problem 1.32).

If $N$ quantities $A^\mu(\mu = 1, 2, \ldots, N)$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to $N$ other quantities $A'_\nu(\nu = 1, 2, \ldots, N)$ in another coordinate system $(x'^1, x'^2, \ldots, x'^N)$ by the transformation equations

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A_\nu \tag{1.96}$$

they are called components of a covariant vector or covariant tensor of the first rank or first order.

One can show easily that velocity and acceleration are contravariant vectors and that the gradient of a scalar field is a covariant vector (Problem 1.33).

Instead of speaking of a tensor whose components are $A^\mu$ or $A_\mu$ we shall simply refer to the tensor $A^\mu$ or $A_\mu$.

## Tensors of second rank

From two contravariant vectors $A^\mu$ and $B^\nu$ we may form the $N^2$ quantities $A^\mu B^\nu$. This is known as the outer product of tensors. These $N^2$ quantities form the components of a contravariant tensor of the second rank: any aggregate of $N^2$ quantities $T^{\mu\nu}$ which, under a coordinate change, transform like the product of two contravariant vectors

$$T^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} T'_{\alpha\beta}, \tag{1.97}$$

is a contravariant tensor of rank two. We may also form a covariant tensor of rank two from two covariant vectors, which transforms according to the formula

$$T_{\mu\nu} = \frac{\partial x'\alpha}{\partial x^{\mu}} \frac{\partial x'\beta}{\partial x^{\nu}} T'_{\alpha\beta}. \tag{1.98}$$

Similarly, we can form a mixed tensor $T^{\mu}{}_{\nu}$ of order two that transforms as follows:

$$T^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'\alpha} \frac{\partial x'\beta}{\partial x^{\nu}} T'^{\alpha}{}_{\beta}. \tag{1.99}$$

We may continue this process and multiply more than two vectors together, taking care that their indexes are all different. In this way we can construct tensors of higher rank. The total number of free indexes of a tensor is its rank (or order).

In a Cartesian coordinate system, the distinction between the contravariant and the covariant tensors vanishes. This can be illustrated with the velocity and gradient vectors. Velocity and acceleration are contravariant vectors, they are represented in terms of components in the directions of coordinate increase; the gradient vector is a covariant vector and it is represented in terms of components in the directions orthogonal to the constant coordinate surfaces. In a Cartesian coordinate system, the coordinate direction $x^{\mu}$ coincides with the direction orthogonal to the constant-$x^{\mu}$ surface, hence the distinction between the covariant and the contravariant vectors vanishes. In fact, this is the essential difference between contravariant and covariant tensors: a covariant tensor is represented by components in directions orthogonal to like constant coordinate surface, and a contravariant tensor is represented by components in the directions of coordinate increase.

If two tensors have the same contravariant rank and the same covariant rank, we say that they are of the same type.

## Basic operations with tensors

(1) Equality: Two tensors are said to be equal if and only if they have the same covariant rank and the same contravariant rank, and every component of one is equal to the corresponding component of the other:

$$A^{\alpha\beta}{}_{\mu} = B^{\alpha\beta}{}_{\mu}.$$

(2) Addition (subtraction): The sum (difference) of two or more tensors of the same type and rank is also a tensor of the same type and rank. Addition of tensors is commutative and associative.

(3) Outer product of tensors: The product of two tensors is a tensor whose rank is the sum of the ranks of the given two tensors. This product involves ordinary multiplication of the components of the tensor and it is called the outer product. For example, $A_\mu{}^{\nu\alpha} B^\beta{}_\lambda = C_{\mu\lambda}{}^{\nu\alpha\beta}$ is the outer product of $A_\mu{}^{\nu\alpha}$ and $B^\beta{}_\lambda$.

(4) Contraction: If a covariant and a contravariant index of a mixed tensor are set equal, a summation over the equal indices is to be taken according to the summation convention. The resulting tensor is a tensor of rank two less than that of the original tensor. This process is called contraction. For example, if we start with a fourth-order tensor $T^\mu{}_{\nu\rho}{}^\delta$, one way of contracting it is to set $\delta = \rho$, which gives the second rank tensor $T^\mu{}_{\nu\rho}{}^\rho$. We could contract it again to get the scalar $T^\mu{}_{\mu\rho}{}^\rho$.

(5) Inner product of tensors: The inner product of two tensors is produced by contracting the outer product of the tensors. For example, given two tensors $A^{\alpha\beta}{}_\delta$ and $B^\mu{}_\nu$, the outer product is $A^{\alpha\beta}{}_\delta B^\mu{}_\nu$. Setting $\delta = \mu$, we obtain the inner product $A^{\alpha\beta}{}_\mu B^\mu{}_\nu$.

(6) Symmetric and antisymmetric tensors: A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unchanged upon interchange of the indices:

$$A^{\alpha\beta} = A^{\beta\alpha}, A_{\alpha\beta} = A_{\beta\alpha}.$$

A tensor is called anti-symmetric with respect to two contravariant or two covariant indices if its components change sign upon interchange of the indices:

$$A^{\alpha\beta} = -A^{\beta\alpha}, A_{\alpha\beta} = -A_{\beta\alpha}.$$

Symmetry and anti-symmetry can be defined only for similar indices, not when one index is up and the other is down.

## Quotient law

A quantity $Q^{\alpha\ldots}{}_{\mu\ldots}$ with various up and down indexes may or may not be a tensor. We can test whether it is a tensor or not by using the quotient law, which can be stated as follows:

> Suppose it is not known whether a quantity $X$ is a tensor or not.
> If an inner product of $X$ with an arbitrary tensor is a tensor, then
> $X$ is also a tensor.

As an example, let $X = P_{\lambda\mu\nu}$, $A^\lambda$ be an arbitrary contravariant vector, and $A^\lambda P_{\lambda\mu\nu}$ be a tensor, say $Q_{\mu\nu}$: $A^\lambda P_{\lambda\mu\nu} = Q_{\mu\nu}$, then

$$A^\lambda P_{\lambda\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} A'^\gamma P'_{\gamma\alpha\beta}.$$

But

$$A'^\gamma = \frac{\partial x'^\gamma}{\partial x^\lambda} A^\lambda$$

and so

$$A^\lambda P_{\lambda\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x'^\gamma}{\partial x^\lambda} A'^\lambda P'_{\gamma\alpha\beta}.$$

This equation must hold for all values of $A^\lambda$, hence we have, after canceling the arbitrary $A^\lambda$,

$$P_{\lambda\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial x'^\gamma}{\partial x^\lambda} P'_{\gamma\alpha\beta},$$

which shows that $P_{\lambda\mu\nu}$ is a tensor (contravariant tensor of rank 3).

## The line element and metric tensor

So far covariant and contravariant tensors have nothing to do each other except that their product is an invariant:

$$A'_\mu B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\beta} A_\alpha A^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha A^\beta = \delta^\alpha{}_\beta A_\alpha A^\beta = A_\alpha A^\alpha.$$

A space in which covariant and contravariant tensors exist separately is called affine. Physical quantities are independent of the particular choice of the mode of description (that is, independent of the possible choice of contravariance or covariance). Such a space is called a metric space. In a metric space, contravariant and covariant tensors can be converted into each other with the help of the metric tensor $g_{\mu\nu}$. That is, in metric spaces there exists the concept of a tensor that may be described by covariant indices, or by contravariant indices. These two descriptions are now equivalent.

To introduce the metric tensor $g_{\mu\nu}$, let us consider the line element in $V_N$. In rectangular coordinates the line element (the differential of arc length) $ds$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2;$$

there are no cross terms $dx^i dx^j$. In curvilinear coordinates $ds^2$ cannot be represented as a sum of squares of the coordinate differentials. As an example, in spherical coordinates we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

which can be in a quadratic form, with $x^1 = r, x^2 = \theta, x^3 = \phi$.

A generalization to $V_N$ is immediate. We define the line element $ds$ in $V_N$ to be given by the following quadratic form, called the metric form, or metric

$$ds^2 = \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.100}$$

For the special cases of rectangular coordinates and spherical coordinates, we have

$$\tilde{g} = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{g} = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{1.101}$$

In an $N$-dimensional orthogonal coordinate system $g_{\mu\nu} = 0$ for $\mu \neq \nu$. And in a Cartesian coordinate system $g_{\mu\mu} = 1$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$. In the general case of Riemannian space, the $g_{\mu\nu}$ are functions of the coordinates $x^\mu (\mu = 1, 2, \ldots, N)$.

Since the inner product of $g_{\mu\nu}$ and the contravariant tensor $dx^\mu dx^\nu$ is a scalar ($ds^2$, the square of line element), then according to the quotient law $g_{\mu\nu}$ is a covariant tensor. This can be demonstrated directly:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g'_{\alpha\beta} dx'^\alpha dx'^\beta.$$

Now $dx'^\alpha = (\partial x'^\alpha / \partial x^\mu) dx^\mu$, so that

$$g'_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu$$

or

$$\left( g'_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} - g_{\mu\nu} \right) dx^\mu dx^\nu = 0.$$

The above equation is identically zero for arbitrary $dx^\mu$, so we have

$$g_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}, \tag{1.102}$$

which shows that $g_{\mu\nu}$ is a covariant tensor of rank two. It is called the metric tensor or the fundamental tensor.

Now contravariant and covariant tensors can be converted into each other with the help of the metric tensor. For example, we can get the covariant vector (tensor of rank one) $A_\mu$ from the contravariant vector $A^\nu$:

$$A_\mu = g_{\mu\nu} A^\nu. \tag{1.103}$$

Since we expect that the determinant of $g_{\mu\nu}$ does not vanish, the above equations can be solved for $A^\nu$ in terms of the $A_\mu$. Let the result be

$$A^\nu = g^{\nu\mu} A_\mu. \tag{1.104}$$

By combining Eqs. (1.103) and (1.104) we get

$$A_\mu = g_{\mu\nu} g^{\nu\alpha} A_\alpha.$$

Since the equation must hold for any arbitrary $A_\mu$, we have

$$g_{\mu\nu} g^{\nu\alpha} = \delta_\mu{}^\alpha, \tag{1.105}$$

where $\delta_\mu{}^\alpha$ is Kronecker's delta symbol. Thus, $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and vice versa; $g^{\mu\nu}$ is often called the conjugate or reciprocal tensor of $g_{\mu\nu}$. But remember that $g^{\mu\nu}$ and $g_{\mu\nu}$ are the contravariant and covariant components of the same tensor, that is the metric tensor. Notice that the matrix $(g^{\mu\nu})$ is just the inverse of the matrix $(g_{\mu\nu})$.

We can use $g_{\mu\nu}$ to lower any upper index occurring in a tensor, and use $g^{\mu\nu}$ to raise any lower index. It is necessary to remember the position from which the index was lowered or raised, because when we bring the index back to its original site, we do not want to interchange the order of indexes, in general $T^{\mu\nu} \neq T^{\nu\mu}$. Thus, for example

$$A^p{}_q = g^{rp} A_{rq}, A^{pq} = g^{rp} g^{sq} A_{rs}, A^p{}_{rs} = g_{rq} A^{pq}{}_s.$$

## Associated tensors

All tensors obtained from a given tensor by forming an inner product with the metric tensor are called associated tensors of the given tensor. For example, $A^\alpha$ and $A_\alpha$ are associated tensors:

$$A_\alpha = g_{\alpha\beta} A^\beta, \quad A^\alpha = g^{\alpha\beta} A_\beta.$$

## Geodesics in a Riemannian space

In a Euclidean space, the shortest path between two points is a straight line joining the two points. In a Riemannian space, the shortest path between two points, called the geodesic, may be a curved path. To find the geodesic, let us consider a space curve in a Riemannian space given by $x^\mu = f^\mu(t)$ and compute the distance between two points of the curve, which is given by the formula

$$s = \int_P^Q \sqrt{g_{\lambda\mu} dx^\lambda dx^\mu} = \int_{t_1}^{t_2} \sqrt{g_{\lambda\mu} d\dot{x}^\lambda d\dot{x}^\mu} dt, \tag{1.106}$$

where $d\dot{x}^\lambda = dx^\lambda/dt$, and $t$ (a parameter) varies from point to point of the geodesic curve described by the relations which we are seeking. A geodesic joining two points $P$ and $Q$ has a stationary value compared with any other neighboring path that connects $P$ and $Q$. Thus, to find the geodesic we extremalize (1.106), and this leads to the differential equation of the geodesic (Problem 1.37)

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0, \tag{1.107}$$

where $F = \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}$, and $\dot{x} = dx/dt$. Now

$$\frac{\partial F}{\partial x^\gamma} = \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{-1/2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\dot{x}^\alpha \dot{x}^\beta, \quad \frac{\partial F}{\partial \dot{x}^\gamma} = \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{-1/2} 2g_{\alpha\gamma}\dot{x}^\alpha$$

and

$$ds/dt = \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}.$$

Substituting these into (1.107) we obtain

$$\frac{d}{dt}\left(g_{\alpha\gamma}\dot{x}^\alpha \dot{s}^{-1}\right) - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\dot{x}^\alpha \dot{x}^\beta \dot{s}^{-1} = 0, \quad \dot{s} = \frac{ds}{dt}$$

or

$$g_{\alpha\gamma}\ddot{x}^\alpha + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\beta - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\dot{x}^\alpha \dot{x}^\beta = g_{\alpha\gamma}\dot{x}^\alpha \ddot{s}\dot{s}^{-1}.$$

We can simplify this equation by writing

$$\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\beta = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\dot{x}^\alpha \dot{x}^\beta,$$

then we have

$$g_{\alpha\gamma}\ddot{x}^\alpha + [\alpha\beta, \gamma]\dot{x}^\alpha \dot{x}^\beta = g_{\alpha\gamma}\dot{x}^\alpha \ddot{s}\dot{s}^{-1}.$$

We can further simplify this equation by taking arc length as the parameter $t$, then $\dot{s} = 1, \ddot{s} = 0$ and we have

$$g_{\alpha\gamma}\frac{d^2x^\alpha}{ds^2} + [\alpha\beta, \gamma]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \tag{1.108}$$

where the functions

$$[\alpha\beta, \gamma] = \Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) \tag{1.109}$$

are called the Christoffel symbols of the first kind.

Multiplying (1.108) by $g^{\rho\gamma}$, we obtain

$$\frac{d^2x^\rho}{ds^2} + \left\{ \begin{matrix} \rho \\ \alpha\beta \end{matrix} \right\} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{1.110}$$

where the functions

$$\left\{ \begin{matrix} \rho \\ \alpha\beta \end{matrix} \right\} = \Gamma^{\rho}{}_{\alpha\beta} = g^{\rho\gamma}[\alpha\beta, \gamma] \tag{1.111}$$

are the Christoffel symbol of the second kind.

Eq. (1.110) is, of course, a set of $N$ coupled differential equations; they are the equations of the geodesic. In Euclidean spaces, geodesics are straight lines. In a Euclidean space, $g_{\alpha\beta}$ are independent of the coordinates $x^{\mu}$, so that the Christoffel symbols identically vanish, and Eq. (1.110) reduces to

$$\frac{d^2x^{\rho}}{ds^2} = 0$$

with the solution

$$x^{\rho} = a_{\rho}s + b_{\rho},$$

where $a_{\rho}$ and $b_{\rho}$ are constants independent of $s$. This solution is clearly a straight line.

The Christoffel symbols are not tensors. Using the defining Eqs. (1.109) and the transformation of the metric tensor, we can find the transformation laws of the Christoffel symbol. We now give the result, without the mathematical details:

$$\bar{\Gamma}_{\mu\nu,\lambda} = \Gamma_{\alpha\beta,\gamma} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\lambda}} + g_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial x^{\lambda}} \frac{\partial^2 x^{\beta}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}}. \tag{1.112}$$

The Christoffel symbols are not tensors because of the presence of the second term on the right hand side.

## Covariant differentiation

We have seen that a covariant vector is transformed according to the formula

$$\bar{A}_{\mu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} A_{\nu},$$

where the coefficients are functions of the coordinates, and so vectors at different points transform differently. Because of this fact, $dA_{\mu}$ is not a vector, since it is the difference of vectors located at two (infinitesimally separated) points. We can verify this directly:

$$\frac{\partial \bar{A}_{\mu}}{\partial \bar{x}^{\gamma}} = \frac{\partial A_{\nu}}{\partial x^{\beta}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\gamma}} + A_{\nu} \frac{\partial^2 x^{\nu}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\gamma}}, \tag{1.113}$$



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which shows that $\partial A/\partial x^\beta$ are not the components of a tensor because of the second term on the right hand side. The same also applies to the differential of a contravariant vector. But we can construct a tensor by the following device.

From Eq. (1.111) we have

$$\bar{\Gamma}^\alpha{}_{\mu_\gamma} = \Gamma^\rho{}_{\sigma\tau} \frac{\partial x^\sigma}{\partial \bar{x}^\mu} \frac{\partial x^\tau}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^\rho} + \frac{\partial^2 x^\sigma}{\partial \bar{x}^\mu \partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma}. \tag{1.114}$$

Multiplying (1.114) by $\bar{A}_\alpha$ and subtracting from (1.113), we obtain

$$\frac{\partial \bar{A}_\mu}{\partial \bar{x}^\gamma} - \bar{A}_\alpha \bar{\Gamma}^\alpha{}_{\mu\gamma} = \left( \frac{\partial A_\alpha}{\partial x^\beta} - A_\rho \Gamma^\rho{}_{\alpha\beta} \right) \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\gamma}. \tag{1.115}$$

If we define

$$A_{\alpha;\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - A_\rho \Gamma^\rho{}_{\alpha\beta}, \tag{1.116}$$

then (1.115) can be rewritten as

$$\bar{A}_{\mu;\gamma} = A_{\alpha;\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\gamma},$$

which shows that $A_{\alpha;\beta}$ is a covariant tensor of rank 2. This tensor is called the covariant derivative of $A_\alpha$ with respect to $x^\beta$. The semicolon denotes covariant differentiation. In a Cartesian coordinate system, the Christoffel symbols vanish, and so covariant differentiation reduces to ordinary differentiation.

The contravariant derivative is found by raising the index which denotes differentiation:

$$A^{\mu;\sigma} = g^{\sigma\alpha} A^\mu{}_{;\alpha}. \tag{1.117}$$

We can similarly determine the covariant derivative of a tensor of arbitrary rank. In doing so we find the following simple rule helps greatly:

> *To obtain the covariant derivative of the tensor $T^{\cdots}_{\cdots}$ with respect to $x^\mu$, we add to the ordinary derivative $\partial T^{\cdots}_{\cdots}/\partial x^\mu$ for each covariant index $\nu(T^{\cdots}_{\cdots\nu.})$ a term $-\Gamma^\alpha{}_{\mu\nu} T^{\cdots}_{\cdots\alpha.}$, and for each contravariant index $\nu(T^{\cdots\nu\cdots}_{\cdots})$ a term $+\Gamma^\alpha{}_{\nu\mu} T^{\cdots\alpha\cdots}_{\cdots}$.*

Thus,

$$T_{\mu\nu;\alpha} = \frac{\partial T_{\mu\nu}}{\partial x^\alpha} - \Gamma^\beta{}_{\mu\alpha} T_{\beta\nu} - \Gamma^\beta{}_{\nu\alpha} T_{\mu\beta},$$

$$T^\mu{}_{\nu;\alpha} = \frac{\partial T^\mu{}_\nu}{\partial x^\alpha} - \Gamma^\beta{}_{\nu\alpha} T^\mu{}_\beta + \Gamma^\mu{}_{\beta\alpha} T^\beta{}_\nu.$$

The covariant derivatives of both the metric tensor and the Kronnecker delta are identically zero (Problem 1.38).

## Problems

1.1. Given the vector $\mathbf{A} = (2, 2, -1)$ and $\mathbf{B} = (6, -3, 2)$, determine:
($a$) $6\mathbf{A} - 3\mathbf{B}$, ($b$) $A^2 + B^2$, ($c$) $\mathbf{A} \cdot \mathbf{B}$, ($d$) the angle between $\mathbf{A}$ and $\mathbf{B}$, ($e$) the direction cosines of $\mathbf{A}$, ($f$) the component of $\mathbf{B}$ in the direction of $\mathbf{A}$.

1.2. Find a unit vector perpendicular to the plane of $\mathbf{A} = (2, -6, -3)$ and $\mathbf{B} = (4, 3, -1)$.

1.3. Prove that:
($a$) the median to the base of an isosceles triangle is perpendicular to the base; ($b$) an angle inscribed in a semicircle is a right angle.

1.4. Given two vectors $\mathbf{A} = (2, 1, -1)$, $\mathbf{B} = (1, -1, 2)$ find: ($a$) $\mathbf{A} \times \mathbf{B}$, and ($b$) a unit vector perpendicular to the plane containing vectors $\mathbf{A}$ and $\mathbf{B}$.

1.5. Prove: ($a$) the law of sines for plane triangles, and ($b$) Eq. (1.16a).

1.6. Evaluate $(2\hat{e}_1 - 3\hat{e}_2) \cdot [(\hat{e}_1 + \hat{e}_2 - \hat{e}_3) \times (3\hat{e}_1 - \hat{e}_3)]$.

1.7. ($a$) Prove that a necessary and sufficient condition for the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ to be coplanar is that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$.
($b$) Find an equation for the plane determined by the three points $P_1(2, -1, 1)$, $P_2(3, 2, -1)$ and $P_3(-1, 3, 2)$.

1.8. ($a$) Find the transformation matrix for a rotation of new coordinate system through an angle $\phi$ about the $x_3(= z)$-axis.
($b$) Express the vector $\mathbf{A} = 3\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3$ in terms of the triad $\hat{e}_1'\hat{e}_2'\hat{e}_3'$ where the $x_1'x_2'$ axes are rotated 45° about the $x_3$-axis (the $x_3$- and $x_3'$-axes coinciding).

1.9. Consider the linear transformation $A_i' = \sum_{j=1}^{3} \hat{e}_i' \cdot \hat{e}_j A_j = \sum_{j=1}^{3} \lambda_{ij} A_j$. Show, using the fact that the magnitude of the vector is the same in both systems, that

$$\sum_{i=1}^{3} \lambda_{ij}\lambda_{ik} = \delta_{jk} \quad (j, k = 1, 2, 3).$$

1.10. A curve $C$ is defined by the parametric equation

$$\mathbf{r}(u) = x_1(u)\hat{e}_1 + x_2(u)\hat{e}_2 + x_3(u)\hat{e}_3,$$

where $u$ is the arc length of C measured from a fixed point on $C$, and $\mathbf{r}$ is the position vector of any point on $C$; show that:
($a$) $d\mathbf{r}/du$ is a unit vector tangent to $C$;
($b$) the radius of curvature of the curve $C$ is given by

$$\rho = \left[\left(\frac{d^2x_1}{du^2}\right)^2 + \left(\frac{d^2x_2}{du^2}\right)^2 + \left(\frac{d^2x_3}{du^2}\right)^2\right]^{-1/2}.$$



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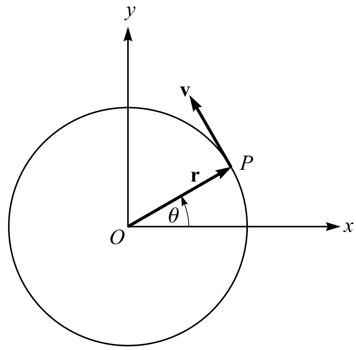


Figure 1.22. Motion on a circle.

1.11. (*a*) Show that the acceleration **a** of a particle which travels along a space curve with velocity **v** is given by

$$\mathbf{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N},$$

where $\hat{T}$, $\hat{N}$, and $\rho$ are as defined in the text.

(*b*) Consider a particle $P$ moving on a circular path of radius **r** with constant angular speed $\omega = d\theta/dt$ (Fig. 1.22). Show that the acceleration **a** of the particle is given by

$$\mathbf{a} = -\omega^2\mathbf{r}.$$

1.12. A particle moves along the curve $x_1 = 2t^2, x_2 = t^2 - 4t, x_3 = 3t - 5$, where $t$ is the time. Find the components of the particle's velocity and acceleration at time $t = 1$ in the direction $\hat{e}_1 - 3\hat{e}_2 + 2\hat{e}_3$.

1.13. (*a*) Find a unit vector normal to the surface $x_1^2 + x_2^2 - x_3 = 1$ at the point $P(1,1,1)$.

(*b*) Find the directional derivative of $\phi = x_1^2 x_2 x_3 + 4x_1 x_3^2$ at $(1, -2, -1)$ in the direction $2\hat{e}_1 - \hat{e}_2 - 2\hat{e}_3$.

1.14. Consider the ellipse given by $r_1 + r_2 =$ const. (Fig. 1.23). Show that $r_1$ and $r_2$ make equal angles with the tangent to the ellipse.

1.15. Find the angle between the surfaces $x_1^2 + x_2^2 + x_3^2 = 9$ and $x_3 = x_1^2 + x_2^2 - 3$. at the point $(2, -1, 2)$.

1.16. (*a*) If $f$ and $g$ are differentiable scalar functions, show that

$$\nabla(fg) = f\nabla g + g\nabla f.$$

(*b*) Find $\nabla r$ if $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

(*c*) Show that $\nabla r^n = nr^{n-2}\mathbf{r}$.

1.17. Show that:

(*a*) $\nabla \cdot (\mathbf{r}/r^3) = 0$. Thus the divergence of an inverse-square force is zero.



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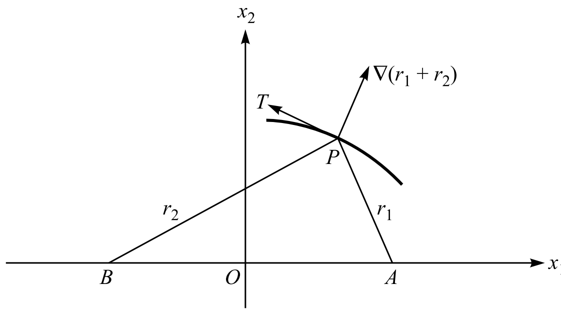


Figure 1.23.

($b$) If $f$ is a differentiable function and $\mathbf{A}$ is a differentiable vector function, then

$$\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}).$$

1.18. ($a$) What is the divergence of a gradient?
($b$) Show that $\nabla^2(1/\mathbf{r}) = 0$.
($c$) Show that $\mathbf{r} \cdot (\nabla \cdot \mathbf{r}) \neq (\mathbf{r}\nabla)\mathbf{r}$.

1.19 Given $\nabla \cdot \mathbf{E} = 0, \nabla \cdot \mathbf{H} = 0, \nabla \times \mathbf{E} = -\partial H/\partial t, \nabla \times \mathbf{H} = \partial E/\partial t$, show that $\mathbf{E}$ and $\mathbf{H}$ satisfy the wave equation $\nabla^2 u = \partial^2 u/\partial t^2$.
The given equations are related to the source-free Maxwell's equations of electromagnetic theory, $\mathbf{E}$ and $\mathbf{H}$ are the electric field and magnetic field intensities.

1.20. ($a$) Find constants $a$, $b$, $c$ such that

$$\mathbf{A} = (x_1 + 2x_2 + ax_3)\hat{e}_1 + (bx_1 - 3x_2 - x_3)\hat{e}_2 + (4x_1 + cx_2 + 2x_3)\hat{e}_3$$

is irrotational.
($b$) Show that $\mathbf{A}$ can be expressed as the gradient of a scalar function.

1.21. Show that a cylindrical coordinate system is orthogonal.

1.22. Find the volume element $dV$ in: (a) cylindrical and (b) spherical coordinates.
Hint: The volume element in orthogonal curvilinear coordinates is

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 = \left|\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)}\right| du_1 du_2 du_3.$$

1.23. Evaluate the integral $\int_{(0,1)}^{(1,2)} (x^2 - y)dx + (y^2 + x)dy$ along
($a$) a straight line from (0, 1) to (1, 2);
($b$) the parabola $x = t, y = t^2 + 1$;
(c) straight lines from (0, 1) to (1, 1) and then from (1, 1) to (1, 2).

1.24. Evaluate the integral $\int_{(0,0)}^{(1,1)} (x^2 + y^2)dx$ along (see Fig. 1.24):
($a$) the straight line $y = x$,
($b$) the circle arc of radius 1 $(x - 1)^2 + y^2 = 1$.

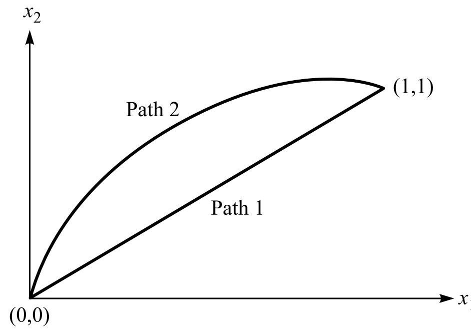


Figure 1.24. Paths for a path integral.

1.25. Evaluate the surface integral $\int_S \mathbf{A} \cdot d\mathbf{a} = \int_S \mathbf{A} \cdot \hat{n} da$, where $\mathbf{A} = x_1 x_2 \hat{e}_1 - x_1^2 \hat{e}_2 + (x_1 + x_2)\hat{e}_3$, $S$ is that portion of the plane $2x_1 + 2x_2 + x_3 = 6$ included in the first octant.

1.26. Verify Gauss' theorem for $\mathbf{A} = (2x_1 - x_3)\hat{e}_1 + x_1^2 x_2 \hat{e}_2 - x_1 x_3^2 \hat{e}_3$ taken over the region bounded by $x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1, x_3 = 0, x_3 = 1$.

1.28 Show that the electrostatic field intensity $E(\mathbf{r})$ of a point charge $Q$ at the origin has an inverse-square dependence on $\mathbf{r}$.

1.28. Show, by using Stokes' theorem, that the gradient of a scalar field is irrotational:

$$\nabla \times (\nabla \phi(\mathbf{r})) = 0.$$

1.29. Verify Stokes' theorem for $\mathbf{A} = (2x_1 - x_2)\hat{e}_1 - x_2 x_3^2 \hat{e}_2 - x_2^2 x_3 \hat{e}_3$, where $S$ is the upper half surface of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ and $\Gamma$ is its boundary (a circle in the $x_1 x_2$ plane of radius 1 with its center at the origin).

1.30. Find the area of the ellipse $x_1 = a\cos\theta, x_2 = b\sin\theta$.

1.31. Show that $\int_S \mathbf{r} \times \hat{n} da = 0$, where $S$ is a closed surface which encloses a volume $V$.

1.33. Starting with Eq. (1.95), express $A'^{\nu}$ in terms of $A^{\mu}$.

1.33. Show that velocity and acceleration are contravariant vectors and that the gradient of a scalar field is a covariant vector.

1.34. The Cartesian components of the acceleration vector are

$$a_x = \frac{d^2 x}{dt^2}, \qquad a_y = \frac{d^2 y}{dt^2}, \qquad a_z = \frac{d^2 z}{dt^2}.$$

Find the component of the acceleration vector in the spherical polar coordinates.

1.35. Show that the property of symmetry (or anti-symmetry) with respect to indexes of a tensor is invariant under coordinate transformation.

1.36. A covariant tensor has components $xy, 2y - z^2, xz$ in rectangular coordinates, find its covariant components in spherical coordinates.

1.37. Prove that a necessary condition that $I = \int_{t_P}^{t_Q} F(t, x, \dot{x})dt$ be an extremum (maximum or minimum) is that

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0.$$

1.38. Show that the covariant derivatives of: (*a*) the metric tensor, and (*b*) the Kronecker delta are identically zero.