2

Ordinary differential equations

Physicists have a variety of reasons for studying differential equations: almost all the elementary and numerous of the advanced parts of theoretical physics are posed mathematically in terms of differential equations. We devote three chapters to differential equations. This chapter will be limited to ordinary differential equations that are reducible to a linear form. Partial differential equations and special functions of mathematical physics will be dealt with in Chapters 10 and 7.

A differential equation is an equation that contains derivatives of an unknown function which expresses the relationship we seek. If there is only one independent variable and, as a consequence, total derivatives like dx/dt, the equation is called an ordinary differential equation (ODE). A partial differential equation (PDE) contains several independent variables and hence partial derivatives.

The *order* of a differential equation is the order of the highest derivative appearing in the equation; its *degree* is the power of the derivative of highest order after the equation has been rationalized, that is, after fractional powers of all derivatives have been removed. Thus the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

is of second order and first degree, and

$$\frac{d^3y}{dx^3} = \sqrt{1 + (dy/dx)^3}$$

is of third order and second degree, since it contains the term $(d^3y/dx^3)^2$ after it is rationalized.

https://doi.org/10.1017/CBO9780511755781.003 Published online by Cambridge University Press

A differential equation is said to be *linear* if each term in it is such that the dependent variable or its derivatives occur only once, and only to the first power. Thus

$$\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 0$$

is not linear, but

$$x^3\frac{d^3y}{dx^3} + e^x \sin x \frac{dy}{dx} + y = \ln x$$

is linear. If in a linear differential equation there are no terms independent of y, the dependent variable, the equation is also said to be *homogeneous*; this would have been true for the last equation above if the '$\ln x$' term on the right hand side had been replaced by zero.

A very important property of linear homogeneous equations is that, if we know two solutions y_1 and y_2, we can construct others as linear combinations of them. This is known as the principle of superposition and will be proved later when we deal with such equations.

Sometimes differential equations look unfamiliar. A trivial change of variables can reduce a seemingly impossible equation into one whose type is readily recognizable.

Many differential equations are very difficult to solve. There are only a relatively small number of types of differential equation that can be solved in closed form. We start with equations of first order. A first-order differential equation can always be solved, although the solution may not always be expressible in terms of familiar functions. A solution (or integral) of a differential equation is the relation between the variables, not involving differential coefficients, which satisfies the differential equation. The solution of a differential equation of order n in general involves n arbitrary constants.

First-order differential equations

A differential equation of the general form

$$\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}, \quad \text{or} \quad g(x,y)dy + f(x,y)dx = 0 \tag{2.1}$$

is clearly a first-order differential equation.

Separable variables

If $f(x,y)$ and $g(x,y)$ are reducible to $P(x)$ and $Q(y)$, respectively, then we have

$$Q(y)dy + P(x)dx = 0. \tag{2.2}$$

Its solution is found at once by integrating.

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The reader may notice that dy/dx has been treated as if it were a ratio of dy and dx, that can be manipulated independently. Mathematicians may be unhappy about this treatment. But, if necessary, we can justify it by considering dy and dx to represent small finite changes δy and δx, before we have actually reached the limit where each becomes infinitesimal.

Example 2.1
Consider the differential equation

$$dy/dx = -y^2 e^x.$$

We can rewrite it in the following form $-dy/y^2 = e^x dx$ which can be integrated separately giving the solution

$$1/y = e^x + c,$$

where c is an integration constant.

Sometimes when the variables are not separable a differential equation may be reduced to one in which they are separable by a change of variable. The general form of differential equation amenable to this approach is

$$dy/dx = f(ax + by), \tag{2.3}$$

where f is an arbitrary function and a and b are constants. If we let $w = ax + by$, then $bdy/dx = dw/dx - a$, and the differential equation becomes

$$dw/dx - a = bf(w)$$

from which we obtain

$$\frac{dw}{a + bf(w)} = dx$$

in which the variables are separated.

Example 2.2
Solve the equation

$$dy/dx = 8x + 4y + (2x + y - 1)^2.$$

Solution: Let $w = 2x + y$, then $dy/dx = dw/dx - 2$, and the differential equation becomes

$$dw/dx + 2 = 4w + (w - 1)^2$$

or

$$dw/[4w + (w - 1)^2 - 2] = dx.$$

The variables are separated and the equation can be solved.

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A homogeneous differential equation which has the general form

$$dy/dx = f(y/x) \tag{2.4}$$

may also be reduced, by a change of variable, to one with separable variables. This can be illustrated by the following example:

Example 2.3
Solve the equation

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}.$$

Solution: The right hand side can be rewritten as $(y/x)^2 + (y/x)$, and hence is a function of the single variable

$$v = y/x.$$

We thus use v both for simplifying the right hand side of our equation, and also for rewriting dy/dx in terms of v and x. Now

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}$$

and our equation becomes

$$v + x\frac{dv}{dx} = v^2 + v$$

from which we have

$$\frac{dv}{v^2} = \frac{dx}{x}.$$

Integration gives

$$-\frac{1}{v} = \ln x + c \quad \text{or} \quad x = Ae^{-x/y},$$

where c and A $(= e^{-c})$ are constants.

Sometimes a nearly homogeneous differential equation can be reduced to homogeneous form which can then be solved by variable separation. This can be illustrated by the by the following:

Example 2.4
Solve the equation

$$dy/dx = (y + x - 5)/(y - 3x - 1).$$

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Solution: Our equation would be homogeneous if it were not for the constants -5 and -1 in the numerator and denominator respectively. But we can eliminate them by a change of variable:

$$x' = x + \alpha, \quad y' = y + \beta,$$

where α and β are constants specially chosen in order to make our equation homogeneous:

$$dy'/dx' = (y' + x')/y' - 3x'.$$

Note that $dy'/dx' = dy/dx$. Trivial algebra yields $\alpha = -1, \beta = -4$. Now let $v = y'/x'$, then

$$\frac{dy'}{dx'} = \frac{d}{dx'}(x'v) = v + x'\frac{dv}{dx'}$$

and our equation becomes

$$v + x'\frac{dv}{dx'} = \frac{v+1}{v-3}, \quad \text{or} \quad \frac{v-3}{-v^2+4v+1}dv = \frac{dx'}{x'}$$

in which the variables are separated and the equation can be solved by integration.

Example 2.5
Fall of a skydiver.

Solution: Assuming the parachute opens at the beginning of the fall, there are two forces acting on the parachute: the downward force of gravity mg, and the upward force of air resistance kv^2. If we choose a coordinate system that has $y = 0$ at the earth's surface and increases upward, then the equation of motion of the falling diver, according to Newton's second law, is

$$mdv/dt = -mg + kv^2,$$

where m is the mass, g the gravitational acceleration, and k a positive constant. In general the air resistance is very complicated, but the power-law approximation is useful in many instances in which the velocity does not vary appreciably. Experiments show that for a subsonic velocity up to 300 m/s, the air resistance is approximately proportional to v^2.

The equation of motion is separable:

$$\frac{mdv}{mg - kv^2} = dt$$

or, to make the integration easier

$$\frac{dv}{v^2 - (mg/k)} = -\frac{k}{m}dt.$$

Now

$$\frac{1}{v^2 - (mg/k)} = \frac{1}{(v + v_t)(v - v_t)} = \frac{1}{2v_t}\left(\frac{1}{v - v_t} - \frac{1}{v + v_t}\right),$$

where $v_t^2 = mg/k$. Thus

$$\frac{1}{2v_t}\left(\frac{dv}{v - v_t} - \frac{dv}{v - v_t}\right) = -\frac{k}{m}dt.$$

Integrating yields

$$\frac{1}{2v_t}\ln\left(\frac{v - v_t}{v + v_t}\right) = -\frac{k}{m}t + c,$$

where c is an integration constant.

Solving for v we finally obtain

$$v(t) = \frac{v_t[1 + B\exp(-2gt/v_t)]}{1 - B\exp(-2gt/v_t)},$$

where $B = \exp(2v_t C)$.

It is easy to see that as $t \to \infty$, $\exp(-2gt/v_t) \to 0$, and so $v \to v_t$; that is, if he falls from a sufficient height, the diver will eventually reach a constant velocity given by v_t, the terminal velocity. To determine the constants of integration, we need to know the value of k, which is about 30 kg/m for the earth's atmosphere and a standard parachute.

Exact equations

We may integrate Eq. (2.1) directly if its left hand side is the differential du of some function $u(x, y)$, in which case the solution is of the form

$$u(x, y) = C \tag{2.5}$$

and Eq. (2.1) is said to be exact. A convenient test to see if Eq. (2.1) is exact is does

$$\frac{\partial g(x, y)}{\partial x} = \frac{\partial f(x, y)}{\partial y}. \tag{2.6}$$

To see this, let us go back to Eq. (2.5) and we have

$$d[u(x, y)] = 0.$$

On performing the differentiation we obtain

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0. \tag{2.7}$$

It is a general property of partial derivatives of any well-behaved function that the order of differentiation is immaterial. Thus we have

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right). \tag{2.8}$$

Now if our differential equation (2.1) is of the form of Eq. (2.7), we must be able to identify

$$f(x, y) = \partial u/\partial x \quad \text{and} \quad g(x, y) = \partial u/\partial y. \tag{2.9}$$

Then it follows from Eq. (2.8) that

$$\frac{\partial g(x, y)}{\partial x} = \frac{\partial f(x, y)}{\partial y},$$

which is Eq. (2.6).

Example 2.6

Show that the equation $xdy/dx + (x + y) = 0$ is exact and find its general solution.

Solution: We first write the equation in standard form

$$(x + y)dx + xdy = 0.$$

Applying the test of Eq. (2.6) we notice that

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x + y) = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{\partial x}{\partial x} = 1.$$

Therefore the equation is exact, and the solution is of the form indicated by Eq. (2.7). From Eq. (2.9) we have

$$\partial u/\partial x = x + y, \quad \partial u/\partial y = x,$$

from which it follows that

$$u(x, y) = x^2/2 + xy + h(y), \quad u(x, y) = xy + k(x),$$

where $h(y)$ and $k(x)$ arise from integrating $u(x, y)$ with respect to x and y, respectively. For consistency, we require that

$$h(y) = 0 \quad \text{and} \quad k(x) = x^2/2.$$

Thus the required solution is

$$x^2/2 + xy = c.$$

It is interesting to consider a differential equation of the type

$$g(x, y)\frac{dy}{dx} + f(x, y) = k(x), \tag{2.10}$$

where the left hand side is an exact differential $(d/dx)[u(x, y)]$, and $k(x)$ on the right hand side is a function of x only. Then the solution of the differential equation can be written as

$$u(x, y) = \int k(x)dx. \tag{2.11}$$

Alternatively Eq. (2.10) can be rewritten as

$$g(x, y)\frac{dy}{dx} + [f(x, y) - k(x)] = 0. \tag{2.10a}$$

Since the left hand side of Eq. (2.10) is exact, we have

$$\partial g/\partial x = \partial f/\partial y.$$

Then Eq. (2.10a) is exact as well. To see why, let us apply the test for exactness for Eq. (2.10a) which requires

$$\frac{\partial}{\partial x}[g(x, y)] = \frac{\partial}{\partial y}[f(x, y) - k(x)] = \frac{\partial}{\partial y}[f(x, y)].$$

Thus Eq. (2.10a) satisfies the necessary requirement for being exact. We can thus write its solution as

$$U(x, y) = c,$$

where

$$\frac{\partial U}{\partial y} = g(x, y) \quad \text{and} \quad \frac{\partial U}{\partial x} = f(x, y) - k(x).$$

Of course, the solution $U(x, y) = c$ must agree with Eq. (2.11).

Integrating factors

If a differential equation in the form of Eq. (2.1) is not already exact, it sometimes can be made so by multiplying by a suitable factor, called an integrating factor. Although an integrating factor always exists for each equation in the form of Eq. (2.1), it may be difficult to find it. However, if the equation is linear, that is, if can be written

$$\frac{dy}{dx} + f(x)y = g(x) \tag{2.12}$$

an integrating factor of the form

$$\exp\left(\int f(x)dx\right) \tag{2.13}$$

is always available. It is easy to verify this. Suppose that $R(x)$ is the integrating factor we are looking for. Multiplying Eq. (2.12) by R, we have

$$R\frac{dy}{dx} + Rf(x)y = Rg(x), \quad \text{or} \quad Rdy + Rf(x)ydx = Rg(x)dx.$$

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The right hand side is already integrable; the condition that the left hand side of Eq. (2.12) be exact gives

$$\frac{\partial}{\partial y}[Rf(x)y] = \frac{\partial R}{\partial x},$$

which yields

$$dR/dx = Rf(x), \quad \text{or} \quad dR/R = f(x)dx,$$

and integrating gives

$$\ln R = \int f(x)dx$$

from which we obtain the integrating factor R we were looking for

$$R = \exp\left(\int f(x)dx\right).$$

It is now possible to write the general solution of Eq. (2.12). On applying the integrating factor, Eq. (2.12) becomes

$$\frac{d(ye^F)}{dx} = g(x)e^F,$$

where $F(x) = \int f(x)dx$. The solution is clearly given by

$$y = e^{-F}\left[\int e^F g(x)dx + C\right].$$

Example 2.7

Show that the equation $xdy/dx + 2y + x^2 = 0$ is not exact; then find a suitable integrating factor that makes the equation exact. What is the solution of this equation?

Solution: We first write the equation in the standard form

$$(2y + x^2)dx + xdy = 0;$$

then we notice that

$$\frac{\partial}{\partial y}(2y + x^2) = 2 \quad \text{and} \quad \frac{\partial}{\partial x}x = 1,$$

which indicates that our equation is not exact. To find the required integrating factor that makes our equation exact, we rewrite our equation in the form of Eq. (2.12):

$$\frac{dy}{dx} + \frac{2y}{x} = -x$$

from which we find $f(x) = 1/x$, and so the required integrating factor is

$$\exp\left(\int (1/x)dx\right) = \exp(\ln x) = x.$$

Applying this to our equation gives

$$x^2\frac{dy}{dx} + 2xy + x^3 = 0 \quad \text{or} \quad \frac{d}{dx}\left(x^2y + x^4/4\right) = 0$$

which integrates to

$$x^2y + x^4/4 = c,$$

or

$$y = \frac{c - x^4}{4x^2}.$$

Example 2.8
RL circuits: A typical *RL* circuit is shown in Fig. 2.1. Find the current $I(t)$ in the circuit as a function of time t.

Solution: We need first to establish the differential equation for the current flowing in the circuit. The resistance R and the inductance L are both constant. The voltage drop across the resistance is IR, and the voltage drop across the inductance is LdI/dt. Kirchhoff's second law for circuits then gives

$$L\frac{dI(t)}{dt} + RI(t) = E(t),$$

which is in the form of Eq. (2.12), but with t as the independent variable instead of x and I as the dependent variable instead of y. Thus we immediately have the general solution

$$I(t) = \frac{1}{L}e^{-Rt/L}\int e^{Rt/L}E(t)dt + ke^{-Rt/L},$$

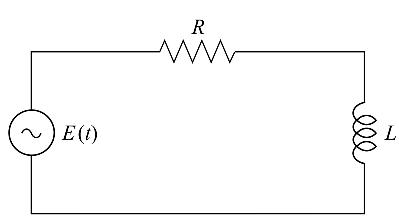

Figure 2.1. *RL* circuit.

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where k is a constant of integration (in electric circuits, C is used for capacitance). Given E this equation can be solved for $I(t)$. If the voltage E is constant, we obtain

$$I(t) = \frac{1}{L} e^{-Rt/L} \left(E \frac{L}{R} e^{-Rt/L} \right) + ke^{-Rt/L} = \frac{E}{R} + ke^{-Rt/L}.$$

Regardless of the value of k, we see that

$$I(t) \to E/R \quad \text{as} \quad t \to \infty.$$

Setting $t = 0$ in the solution, we find

$$k = I(0) - E/R.$$

Bernoulli's equation

Bernoulli's equation is a non-linear first-order equation that occurs occasionally in physical problems:

$$\frac{dy}{dx} + f(x)y = g(x)y^n, \tag{2.14}$$

where n is not necessarily integer.

This equation can be made linear by the substitution $w = y^a$ with α suitably chosen. We find this can be achieved if $\alpha = 1 - n$:

$$w = y^{1-n} \quad \text{or} \quad y = w^{1/(1-n)}.$$

This converts Bernoulli's equation into

$$\frac{dw}{dx} + (1-n)f(x)w = (1-n)g(x),$$

which can be made exact using the integrating factor $\exp(\int (1-n)f(x)dx)$.

Second-order equations with constant coefficients

The general form of the nth-order linear differential equation with constant coefficients is

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_n y = (D^n + p_1 D^{n-1} + \cdots + p_{n-1}D + p_n)y = f(x),$$

where $p_1, p_2, \ldots$ are constants, $f(x)$ is some function of x, and $D \equiv d/dx$. If $f(x) = 0$, the equation is called homogeneous; otherwise it is called a non-homogeneous equation. It is important to note that the symbol D is meaningless unless applied to a function of x and is therefore not a mathematical quantity in the usual sense. D is an operator.

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Many of the differential equations of this type which arise in physical problems are of second order and we shall consider in detail the solution of the equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = (D^2 + aD + b)y = f(t), \tag{2.15}$$

where a and b are constants, and t is the independent variable. As an example, the equation of motion for a mass on a spring is of the form Eq. (2.15), with a representing the friction, c being the constant of proportionality in Hooke's law for the spring, and $f(t)$ some time-dependent external force acting on the mass. Eq. (2.15) can also apply to an electric circuit consisting of an inductor, a resistor, a capacitor and a varying external voltage.

The solution of Eq. (2.15) involves first finding the solution of the equation with $f(t)$ replaced by zero, that is,

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = (D^2 + aD + b)y = 0; \tag{2.16}$$

this is called the reduced or homogeneous equation corresponding to Eq. (2.15).

Nature of the solution of linear equations

We now establish some results for linear equations in general. For simplicity, we consider the second-order reduced equation (2.16). If y_1 and y_2 are independent solutions of (2.16) and A and B are any constants, then

$$D(Ay_1 + By_2) = ADy_1 + BDy_2, \quad D^2(Ay_1 + By_2) = AD^2y_1 + BD^2y_2$$

and hence

$$(D^2 + aD + b)(Ay_1 + By_2) = A(D^2 + aD + b)y_1 + B(D^2 + aD + b)y_2 = 0.$$

Thus $y = Ay_1 + By_2$ is a solution of Eq. (2.16), and since it contains two arbitrary constants, it is the general solution. A necessary and sufficient condition for two solutions y_1 and y_2 to be linearly independent is that the Wronskian determinant of these functions does not vanish:

$$\begin{vmatrix} y_1 & y_2 \\ \dfrac{dy_1}{dt} & \dfrac{dy_2}{dt} \end{vmatrix} \neq 0.$$

Similarly, if $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of the nth-order linear equations, then the general solution is

$$y = A_1y_1 + A_2y_2 + \cdots + A_ny_n,$$

where $A_1, A_2, \ldots, A_n$ are arbitrary constants. This is known as the superposition principle.

General solutions of the second-order equations

Suppose that we can find one solution, $y_p(t)$ say, of Eq. (2.15):

$$(D^2 + aD + b)y_p(t) = f(t). \tag{2.15a}$$

Then on defining

$$y_c(t) = y(t) - y_p(t)$$

we find by subtracting Eq. (2.15a) from Eq. (2.15) that

$$(D^2 + aD + b)y_c(t) = 0.$$

That is, $y_c(t)$ satisfies the corresponding homogeneous equation (2.16), and it is known as the complementary function $y_c(t)$ of non-homogeneous equation (2.15). while the solution $y_p(t)$ is called a particular integral of Eq. (2.15). Thus, the general solution of Eq. (2.15) is given by

$$y(t) = Ay_c(t) + By_p(t). \tag{2.17}$$

Finding the complementary function

Clearly the complementary function is independent of $f(t)$, and hence has nothing to do with the behavior of the system in response to the external applied influence. What it does represent is the free motion of the system. Thus, for example, even without external forces applied, a spring can oscillate, because of any initial displacement and/or velocity. Similarly, had a capacitor already been charged at $t = 0$, the circuit would subsequently display current oscillations even if there is no applied voltage.

In order to solve Eq. (2.16) for $y_c(t)$, we first consider the linear first-order equation

$$a\frac{dy}{dt} + by = 0.$$

Separating the variables and integrating, we obtain

$$y = Ae^{-bt/a},$$

where A is an arbitrary constant of integration. This solution suggests that Eq. (2.16) might be satisfied by an expression of the type

$$y = e^{pt},$$

where p is a constant. Putting this into Eq. (2.16), we have

$$e^{pt}(p^2 + ap + b) = 0.$$

Therefore $y = e^{pt}$ is a solution of Eq. (2.16) if

$$p^2 + ap + b = 0.$$

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This is called the auxiliary (or characteristic) equation of Eq. (2.16). Solving it gives

$$p_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad p_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}. \tag{2.18}$$

We now distinguish between the cases in which the roots are real and distinct, complex or coincident.

(*i*) Real and distinct roots ($a^2 - 4b > 0$)
In this case, we have two independent solutions $y_1 = e^{p_1 t}, y_2 = e^{p_2 t}$ and the general solution of Eq. (2.16) is a linear combination of these two:

$$y = Ae^{p_1 t} + Be^{p_2 t}, \tag{2.19}$$

where A and B are constants.

Example 2.9
Solve the equation $(D^2 - 2D - 3)y = 0$, given that $y = 1$ and $y' = dy/dx = 2$ when $t = 0$.

Solution: The auxiliary equation is $p^2 - 2p - 3 = 0$, from which we find $p = -1$ or $p = 3$. Hence the general solution is

$$y = Ae^{-t} + Be^{3t}.$$

The constants A and B can be determined by the boundary conditions at $t = 0$. Since $y = 1$ when $t = 0$, we have

$$1 = A + B.$$

Now

$$y' = -Ae^{-t} + 3Be^{3t}$$

and since $y' = 2$ when $t = 0$, we have $2 = -A + 3B$. Hence

$$A = 1/4, \quad B = 3/4$$

and the solution is

$$4y = e^{-t} + 3e^{3t}.$$

(*ii*) Complex roots ($a^2 - 4b < 0$)
If the roots p_1, p_2 of the auxiliary equation are imaginary, the solution given by Eq. (2.18) is still correct. In order to give the solutions in terms of real quantities, we can use the Euler relations to express the exponentials. If we let $r = -a/2, is = \sqrt{a^2 - 4b}/2$, then

$$e^{p_1 t} = e^{rt}e^{ist} = e^{rt}[\cos st + i \sin st],$$

$$e^{p_2 t} = e^{rt}e^{ist} = e^{rt}[\cos st - i \sin st]$$

and the general solution can be written as

$$\begin{aligned} y &= Ae^{p_1 t} + Be^{p_2 t} \\ &= e^{rt}[(A+B)\cos st + i(A-B)\sin st] \\ &= e^{rt}[A_0 \cos st + B_0 \sin st] \end{aligned} \tag{2.20}$$

with $A_0 = A + B, B_0 = i(A - B)$.

The solution (2.20) may be expressed in a slightly different and often more useful form by writing $B_0/A_0 = \tan\delta$. Then

$$y = (A_0^2 + B_0^2)^{1/2} e^{rt}(\cos\delta\cos st + \sin\delta\sin st) = Ce^{rt}\cos(st - \delta), \tag{2.20a}$$

where C and δ are arbitrary constants.

Example 2.10

Solve the equation $(D^2 + 4D + 13)y = 0$, given that $y = 1$ and $y' = 2$ when $t = 0$.

Solution: The auxiliary equation is $p^2 + 4p + 13 = 0$, and hence $p = -2 \pm 3i$. The general solution is therefore, from Eq. (2.20),

$$y = e^{-2t}(A_0 \cos 3t + B_0 \sin 3t).$$

Since $y = l$ when $t = 0$, we have $A_0 = 1$. Now

$$y' = -2e^{-2t}(A_0 \cos 3t + B_0 \sin 3t) + 3e^{-2t}(-A_0 \sin 3t + B_0 \cos 3t)$$

and since $y' = 2$ when $t = 0$, we have $2 = -2A_0 + 3B_0$. Hence $B_0 = 4/3$, and the solution is

$$3y = e^{-2t}(3\cos 3t + 4\sin 3t).$$

(*iii*) Coincident roots

When $a^2 = 4b$, the auxiliary equation yields only one value for p, namely $p = \alpha = -a/2$, and hence the solution $y = Ae^{\alpha t}$. This is not the general solution as it does not contain the necessary two arbitrary constants. In order to obtain the general solution we proceed as follows. Assume that $y = ve^{\alpha t}$, where v is a function of t to be determined. Then

$$y' = v'e^{\alpha t} + \alpha v e^{\alpha t}, y'' = v'' e^{\alpha t} + 2\alpha v' e^{\alpha t} + \alpha^2 v e^{\alpha t}.$$

Substituting for y, y', and y'' in the differential equation we have

$$e^{\alpha t}[v'' + 2\alpha v' + \alpha^2 v + a(v' + \alpha v) + bv] = 0$$

and hence

$$v'' + v'(a + 2\alpha) + v(\alpha^2 + a\alpha + b) = 0.$$

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Now

$$\alpha^2 + a\alpha + b = 0, \quad \text{and} \quad a + 2\alpha = 0$$

so that

$$v'' = 0.$$

Hence, integrating gives

$$v = At + B,$$

where A and B are arbitrary constants, and the general solution of Eq. (2.16) is

$$y = (At + B)e^{\alpha t} \tag{2.21}$$

Example 2.11
Solve the equation $(D^2 - 4D + 4)y = 0$ given that $y = 1$ and $Dy = 3$ when $t = 0$.

Solution: The auxiliary equation is $p^2 - 4p + 4 = (p - 2)^2 = 0$ which has one root $p = 2$. The general solution is therefore, from Eq. (2.21)

$$y = (At + B)e^{2t}.$$

Since $y = 1$ when $t = 0$, we have $B = 1$. Now

$$y' = 2(At + B)e^{2t} + Ae^{2t}$$

and since $Dy = 3$ when $t = 0$,

$$3 = 2B + A.$$

Hence $A = 1$ and the solution is

$$y = (t + 1)e^{2t}.$$

Finding the particular integral

The particular integral is a solution of Eq. (2.15) that takes the term $f(t)$ on the right hand side into account. The complementary function is transient in nature, so from a physical point of view, the particular integral will usually dominate the response of the system at large times.

The method of determining the particular integral is to guess a suitable functional form containing arbitrary constants, and then to choose the constants to ensure it is indeed the solution. If our guess is incorrect, then no values of these constants will satisfy the differential equation, and so we have to try a different form. Clearly this procedure could take a long time; fortunately, there are some guiding rules on what to try for the common examples of $f(t)$:

(1) $f(t) =$ a polynomial in t.
If $f(t)$ is a polynomial in t with highest power t^n, then the trial particular integral is also a polynomial in t, with terms up to the same power. Note that the trial particular integral is a power series in t, even if $f(t)$ contains only a single terms At^n.

(2) $f(t) = Ae^{kt}$.
The trial particular integral is $y = Be^{kt}$.

(3) $f(t) = A\sin kt$ or $A\cos kt$.
The trial particular integral is $y = A\sin kt + C\cos kt$. That is, even though $f(t)$ contains only a sine or cosine term, we need both sine and cosine terms for the particular integral.

(4) $f(t) = Ae^{\alpha t}\sin\beta t$ or $Ae^{\alpha t}\cos\beta t$.
The trial particular integral is $y = e^{\alpha t}(B\sin\beta t + C\cos\beta t)$.

(5) $f(t)$ is a polynomial of order n in t, multiplied by e^{kt}.
The trial particular integral is a polynomial in t with coefficients to be determined, multiplied by e^{kt}.

(6) $f(t)$ is a polynomial of order n in t, multiplied by $\sin kt$.
The trial particular integral is $y = \Sigma_{j=0}^{n}(B_j\sin kt + C_j\cos kt)t^j$. Can we try $y = (B\sin kt + C\cos kt)\,\Sigma_{j=0}^{n} D_j t^j$? The answer is no. Do you know why?

If the trial particular integral or part of it is identical to one of the terms of the complementary function, then the trial particular integral must be multiplied by an extra power of t. Therefore, we need to find the complementary function before we try to work out the particular integral. What do we mean by ‘identical in form’? It means that the ratio of their t-dependences is a constant. Thus $-2e^{-t}$ and Ae^{-t} are identical in form, but e^{-t} and e^{-2t} are not.

Particular integral and the operator D $(= d/dx)$

We now describe an alternative method that can be used for finding particular integrals. As compared with the method described in previous section, it involves less guesswork as to what the form of the solution is, and the constants multiplying the functional forms of the answer are obtained automatically. It does, however, require a fair amount of practice to ensure that you are familiar with how to use it.

The technique involves using the differential operator $D \equiv d(\;)/dt$, which is an interesting and simple example of a linear operator without a matrix representation. It is obvious that D obeys the relevant laws of operator algebra: suppose f and g are functions of t, and a is a constant, then

(i) $D(f+g) = Df + Dg$ (distributive);
(ii) $Daf = aDf$ (commutative);
(iii) $D^nD^mf = D^{n+m}f$ (index law).

We can form a polynomial function of D and write

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

so that

$$F(D)f(t) = a_0 D^n f + a_1 D^{n-1} f + \cdots + a_{n-1} Df + a_n f$$

and we can interpret D^{-1} as follows

$$D^{-1} Df(t) = f(t)$$

and

$$\int (Df)dt = f.$$

Hence D^{-1} indicates the operation of integration (the inverse of differentiation). Similarly $D^{-m}f$ means 'integrate $f(t)m$ times'.

These properties of the linear operator D can be used to find the particular integral of Eq. (2.15):

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \left(D^2 + aD + b\right)y = f(t)$$

from which we obtain

$$y = \frac{1}{D^2 + aD + b} f(t) = \frac{1}{F(D)} f(t), \tag{2.22}$$

where

$$F(D) = D^2 + aD + b.$$

The trouble with Eq. (2.22) is that it contains an expression involving Ds in the denominator. It requires a fair amount of practice to use Eq. (2.22) to express y in terms of conventional functions. For this, there are several rules to help us.

Rules for D operators

Given a power series of D

$$G(D) = a_0 + a_1 D + \cdots + a_n D^n + \cdots$$

and since $D^n e^{\alpha t} = \alpha^n e^{\alpha t}$, it follows that

$$G(D)e^{\alpha t} = (a_0 + a_1 D + \cdots + a_n D^n + \cdots)e^{\alpha t} = G(\alpha)e^{\alpha t}.$$

Thus we have

Rule (a): $G(D)e^{\alpha t} = G(\alpha)e^{\alpha t}$ provided $G(\alpha)$ is convergent.

When $G(D)$ is the expansion of $1/F(D)$ this rule gives

$$\frac{1}{F(D)} e^{\alpha t} = \frac{1}{F(\alpha)} e^{\alpha t} \quad \text{provided } F(\alpha) \neq 0.$$

Now let us operate $G(D)$ on a product function $e^{\alpha t}V(t)$:

$$G(D)[e^{\alpha t}V(t)] = [G(D)e^{\alpha t}]V(t) + e^{\alpha t}[G(D)V(t)]$$
$$= e^{\alpha t}[G(\alpha) + G(D)]V(t) = e^{\alpha t}G(D+\alpha)[V(t)].$$

That is, we have

Rule (b): $G(D)[e^{\alpha t}V(t)] = e^{\alpha t}G(D+\alpha)[V(t)]$.

Thus, for example

$$D^2[e^{\alpha t}t^2] = e^{\alpha t}(D+\alpha)^2[t^2].$$

Rule (c): $G(D^2)\sin kt = G(-k^2)\sin kt$.

Thus, for example

$$\frac{1}{D^2}(\sin 3t) = -\frac{1}{9}\sin 3t.$$

Example 2.12 Damped oscillations (Fig. 2.2)

Suppose we have a spring of natural length L (that is, in its unstretched state). If we hang a ball of mass m from it and leave the system in equilibrium, the spring stretches an amount d, so that the ball is now $L + d$ from the suspension point. We measure the vertical displacement of the ball from this static equilibrium point. Thus, $L + d$ is $y = 0$, and y is chosen to be positive in the downward direction, and negative upward. If we pull down on the ball and then release it, it oscillates up and down about the equilibrium position. To analyze the oscillation of the ball, we need to know the forces acting on it:

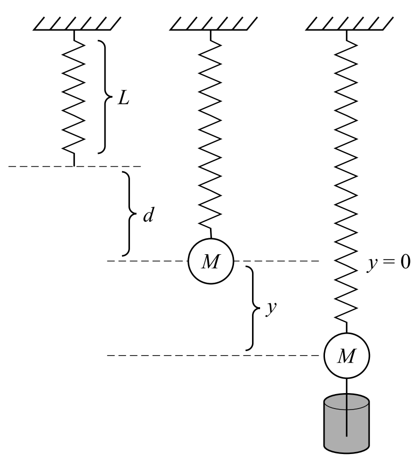

Figure 2.2. Damped spring system.

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(1) the downward force of gravity, mg:
(2) the restoring force ky which always opposes the motion (Hooke's law), where k is the spring constant of the spring. If the ball is pulled down a distance y from its static equilibrium position, this force is $-k(d+y)$.

Thus, the total net force acting on the ball is

$$mg - k(d+y) = mg - kd - ky.$$

In static equilibrium, $y = 0$ and all forces balances. Hence

$$kd = mg$$

and the net force acting on the spring is just $-ky$; and the equation of motion of the ball is given by Newton's second law of motion:

$$m\frac{d^2y}{dt^2} = -ky,$$

which describes free oscillation of the ball. If the ball is connected to a dashpot (Fig. 2.2), a damping force will come into play. Experiment shows that the damping force is given by $-bdy/dt$, where the constant b is called the damping constant. The equation of motion of the ball now is

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} \quad \text{or} \quad y'' + \frac{b}{m}y' + \frac{k}{m}y = 0.$$

The auxiliary equation is

$$p^2 + \frac{b}{m}p + \frac{k}{m} = 0$$

with roots

$$p_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2 - 4km}, \qquad p_2 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2 - 4km}.$$

We now have three cases, resulting in quite different motions of the oscillator.

Case 1 $b^2 - 4km > 0$ (overdamping)
The solution is of the form

$$y(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t}.$$

Now, both b and k are positive, so

$$\frac{1}{2m}\sqrt{b^2 - 4km} < \frac{b}{2m}$$

and accordingly

$$p_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2 - 4km} < 0.$$

Obviously $p_2 < 0$ also. Thus, $y(t) \to 0$ as $t \to \infty$. This means that the oscillation dies out with time and eventually the mass will assume the static equilibrium position.

Case 2 $b^2 - 4km = 0$ (critical damping)
The solution is of the form

$$y(t) = e^{-bt/2m}(c_1 + c_2 t).$$

As both b and m are positive, $y(t) \to 0$ as $t \to \infty$ as in case 1. But c_1 and c_2 play a significant role here. Since $e^{-bt/2m} \neq 0$ for finite t, $y(t)$ can be zero only when $c_1 + c_2 t = 0$, and this happens when

$$t = -c_1/c_2.$$

If the number on the right is positive, the mass passes through the equilibrium position $y = 0$ at that time. If the number on the right is negative, the mass never passes through the equilibrium position.

It is interesting to note that $c_1 = y(0)$, that is, c_1 measures the initial position. Next, we note that

$$y'(0) = c_2 - bc_1/2m, \quad \text{or} \quad c_2 = y'(0) + by(0)/2m.$$

Case 3 $b^2 - 4km < 0$ (underdamping)
The auxiliary equation now has complex roots

$$p_1 = -\frac{b}{2m} + \frac{i}{2m}\sqrt{4km - b^2}, \quad p_2 = -\frac{b}{2m} - \frac{i}{2m}\sqrt{4km - b^2}$$

and the solution is of the form

$$y(t) = e^{-bt/2m}\left[c_1 \cos\left(\sqrt{4km - b^2}\right)\frac{t}{2m} + c_2 \sin\left(\sqrt{4km - b^2}\,\frac{t}{2m}\right)\right],$$

which can be rewritten as

$$y(t) = ce^{-bt/2m}\cos(\omega t - \alpha),$$

where

$$c = \sqrt{c_1^2 + c_2^2}, \alpha = \tan^{-1}\left(\frac{c_2}{c_1}\right), \quad \text{and} \quad \omega = \sqrt{4km - b^2}/2m.$$

As in case 2, $e^{-bt/2m} \to 0$ as $t \to \infty$, and the oscillation gradually dies down to zero with increasing time. As the oscillator dies down, it oscillates with a frequency $\omega/2\pi$. But the oscillation is not periodic.

https://doi.org/10.1017/CBO9780511755781.003 Published online by Cambridge University Press

The Euler linear equation

The linear equation with variable coefficients

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1} x \frac{dy}{dx} + p_n y = f(x), \tag{2.23}$$

in which the derivative of the jth order is multiplied by x^j and by a constant, is known as the Euler or Cauchy equation. It can be reduced, by the substitution $x = e^t$, to a linear equation with constant coefficients with t as the independent variable. Now if $x = e^t$, then $dx/dt = x$, and

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}, \quad \text{or} \quad x\frac{dy}{dx} = \frac{dy}{dt}$$

and

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

or

$$x\frac{d^2 y}{dx^2} = \frac{1}{x}\frac{d^2 y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left(\frac{1}{x}\right) = \frac{1}{x}\frac{d^2 y}{dx^2} - \frac{1}{x}\frac{dy}{dt}$$

and hence

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt} = \frac{d}{dt}\left(\frac{dy}{dt} - 1\right)y.$$

Similarly

$$x^3 \frac{d^3 y}{dx^3} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right)y,$$

and

$$x^n \frac{d^n y}{dx^n} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right)\cdots\left(\frac{d}{dt} - n + 1\right)y.$$

Substituting for $x^j(d^j y/dx^j)$ in Eq. (2.23) the equation transforms into

$$\frac{d^n y}{dt^n} + q_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + q_{n-1}\frac{dy}{dt} + q_n y = f(e^t)$$

in which $q_1, q_2, \ldots, q_n$ are constants.

Example 2.13

Solve the equation

$$x^2 \frac{d^2 y}{dx^2} + 6x\frac{dy}{dx} + 6y = \frac{1}{x^2}.$$

Solution: Put $x = e^t$, then

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Substituting these in the equation gives

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^t.$$

The auxiliary equation $p^2 + 5p + 6 = (p+2)(p+3) = 0$ has two roots: $p_1 = -2$, $p_2 = 3$. So the complementary function is of the form $y_c = Ae^{-2t} + Be^{-3t}$ and the particular integral is

$$y_p = \frac{1}{(D+2)(D+3)}e^{-2t} = te^{-2t}.$$

The general solution is

$$y = Ae^{-2t} + Be^{-3t} + te^{-2t}.$$

The Euler equation is a special case of the general linear second-order equation

$$D^2y + p(x)Dy + q(x)y = f(x),$$

where $p(x)$, $q(x)$, and $f(x)$ are given functions of x. In general this type of equation can be solved by series approximation methods which will be introduced in next section, but in some instances we may solve it by means of a variable substitution, as shown by the following example:

$$D^2y + (4x - x^{-1})Dy + 4x^2y = 0,$$

where

$$p(x) = (4x - x^{-1}), \quad q(x) = 4x^2, \quad \text{and} \quad f(x) = 0.$$

If we let

$$x = z^{1/2}$$

the above equation is transformed into the following equation with constant coefficients:

$$D^2y + 2Dy + y = 0,$$

which has the solution

$$y = (A + Bz)e^{-z}.$$

Thus the general solution of the original equation is $y = (A + Bx^2)e^{-x^2}$.

https://doi.org/10.1017/CBO9780511755781.003 Published online by Cambridge University Press

Solutions in power series

In many problems in physics and engineering, the differential equations are of such a form that it is not possible to express the solution in terms of elementary functions such as exponential, sine, cosine, etc.; but solutions can be obtained as convergent infinite series. What is the basis of this method? To see it, let us consider the following simple second-order linear differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Now assuming the solution is given by $y = a_0 + a_1x + a_2x^2 + \cdots$, we further assume the series is convergent and differentiable term by term for sufficiently small x. Then

$$dy/dx = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

and

$$d^2y/dx^2 = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots.$$

Substituting the series for y and d^2y/dx^2 in the given differential equation and collecting like powers of x yields the identity

$$(2a_2 + a_0) + (2\times 3a_3 + a_1)x + (3\times 4a_4 + a_2)x^2 + \cdots = 0.$$

Since if a power series is identically zero all of its coefficients are zero, equating to zero the term independent of x and coefficients of x, $x^2, \ldots,$ gives

$$\begin{aligned} 2a_2 + a_0 &= 0, & 4\times 5a_5 + a_3 &= 0,\\ 2\times 3a_3 + a_1 &= 0, & 5\times 6a_6 + a_4 &= 0,\\ 3\times 4a_4 + a_2 &= 0, & \cdots \end{aligned}$$

and it follows that

$$a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{2\times 3} = -\frac{a_1}{3!}, \quad a_4 = -\frac{a_2}{3\times 4} - \frac{a_0}{4!}$$

$$a_5 = -\frac{a_3}{4\times 5} = \frac{a_1}{5!}, \quad a_6 = -\frac{a_4}{5\times 6} = -\frac{a_0}{6!}, \ldots.$$

The required solution is

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right);$$

you should recognize this as equivalent to the usual solution $y = a_0\cos x + a_1\sin x$, a_0 and a_1 being arbitrary constants.

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Ordinary and singular points of a differential equation

We shall concentrate on the linear second-order differential equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{2.24}$$

which plays a very important part in physical problems, and introduce certain definitions and state (without proofs) some important results applicable to equations of this type. With some small modifications, these are applicable to linear equation of any order. If both the functions P and Q can be expanded in Taylor series in the neighborhood of $x = \alpha$, then Eq. (2.24) is said to possess an ordinary point at $x = \alpha$. But when either of the functions P or Q does not possess a Taylor series in the neighborhood of $x = \alpha$, Eq. (2.24) is said to have a singular point at $x = \alpha$. If

$$P = \lambda(x)/(x-\alpha) \quad \text{and} \quad Q = \mu(x)/(x-\alpha)^2$$

and $\lambda(x)$ and $\mu(x)$ can be expanded in Taylor series near $x = \alpha$. In such cases, $x = \alpha$ is a singular point but the singularity is said to be regular.

Frobenius and Fuchs theorem

Frobenius and Fuchs showed that:

(1) If $P(x)$ and $Q(x)$ are regular at $x = \alpha$, then the differential equation (2.24) possesses two distinct solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda (x-\alpha)^\lambda \quad (a_0 \neq 0). \tag{2.25}$$

(2) If $P(x)$ and $Q(x)$ are singular at $x = \alpha$, but $(x-\alpha)P(x)$ and $(x-\alpha)^2 Q(x)$ are regular at $x = \alpha$, then there is at least one solution of the differential equation (2.24) of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda (x-\alpha)^{\lambda+\rho} \quad (a_0 \neq 0), \tag{2.26}$$

where ρ is some constant, which is valid for $|x-\alpha| < \beta$ whenever the Taylor series for $\lambda(x)$ and $\mu(x)$ are valid for these values of x.

(3) If $P(x)$ and $Q(x)$ are irregular singular at $x = \alpha$ (that is, $\lambda(x)$ and $\mu(x)$ are singular at $x = \alpha$), then regular solutions of the differential equation (2.24) may not exist.

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The proofs of these results are beyond the scope of the book, but they can be found, for example, in E. L. Ince's *Ordinary Differential Equations*, Dover Publications Inc., New York, 1944.

The first step in finding a solution of a second-order differential equation relative to a regular singular point $x = \alpha$ is to determine possible values for the index ρ in the solution (2.26). This is done by substituting series (2.26) and its appropriate differential coefficients into the differential equation and equating to zero the resulting coefficient of the lowest power of $x - \alpha$.This leads to a quadratic equation, called the indicial equation, from which suitable values of ρ can be found. In the simplest case, these values of ρ will give two different series solutions and the general solution of the differential equation is then given by a linear combination of the separate solutions. The complete procedure is shown in Example 2.14 below.

Example 2.14
Find the general solution of the equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

Solution: The origin is a regular singular point and, writing $y = \Sigma_{\lambda=0}^{\infty} a_\lambda x^{\lambda+\rho} (a_0 \neq 0)$ we have

$$dy/dx = \sum_{\lambda=0}^{\infty} a_\lambda(\lambda+\rho)x^{\lambda+\rho-1}, \quad d^2y/dx^2 = \sum_{\lambda=0}^{\infty} a_\lambda(\lambda+\rho)(\lambda+\rho-1)x^{\lambda+\rho-2}.$$

Before substituting in the differential equation, it is convenient to rewrite it in the form

$$\left\{4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right\} + \{y\} = 0.$$

When $a_\lambda x^{\lambda+\rho}$ is substituted for y, each term in the first bracket yields a multiple of $x^{\lambda+\rho-1}$, while the second bracket gives a multiple of $x^{\lambda+\rho}$ and, in this form, the differential equation is said to be arranged according to weight, the weights of the bracketed terms differing by unity. When the assumed series and its differential coefficients are substituted in the differential equation, the term containing the lowest power of x is obtained by writing $y = a_0 x^\rho$ in the first bracket. Since the coefficient of the lowest power of x must be zero and, since $a_0 \neq 0$, this gives the indicial equation

$$4\rho(\rho-1) + 2\rho = 2\rho(2\rho-1) = 0;$$

its roots are $\rho = 0$, $\rho = 1/2$.

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The term in $x^{\lambda+\rho}$ is obtained by writing $y = a_{\lambda+1}x^{\lambda+\rho+1}$ in first bracket and $y = a_\lambda x^{\lambda+\rho}$ in the second. Equating to zero the coefficient of the term obtained in this way we have

$$\{4(\lambda+\rho+1)(\lambda+\rho)+2(\lambda+\rho+1)\}a_{\lambda+1}+a_\lambda = 0,$$

giving, with λ replaced by n,

$$a_{n+1} = -\frac{1}{2(\rho+n+1)(2\rho+2n+1)}a_n.$$

This relation is true for $n = 1, 2, 3, \ldots$ and is called the recurrence relation for the coefficients. Using the first root $\rho = 0$ of the indicial equation, the recurrence relation gives

$$a_{n+1} = \frac{1}{2(n+1)(2n+1)}a_n$$

and hence

$$a_1 = -\frac{a_0}{2}, \quad a_2 = -\frac{a_1}{12} = \frac{a_0}{4!}, \quad a_3 = -\frac{a_2}{30} = -\frac{a_0}{6!}, \ldots.$$

Thus one solution of the differential equation is the series

$$a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + - \cdots\right).$$

With the second root $\rho = 1/2$, the recurrence relation becomes

$$a_{n+1} = -\frac{1}{(2n+3)(2n+2)}a_n.$$

Replacing a_0 (which is arbitrary) by b_0, this gives

$$a_1 = -\frac{b_0}{3\times 2} = -\frac{b_0}{3!}, \quad a_2 = -\frac{a_1}{5\times 4} = \frac{b_0}{5!}, \quad a_3 = -\frac{a_2}{7\times 6} = -\frac{b_0}{7!}, \ldots.$$

and a second solution is

$$b_0 x^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + - \cdots\right).$$

The general solution of the equation is a linear combination of these two solutions.

Many physical problems require solutions which are valid for large values of the independent variable x. By using the transformation $x = 1/t$, the differential equation can be transformed into a linear equation in the new variable t and the solutions required will be those valid for small t.

In Example 2.14 the indicial equation has two distinct roots. But there are two other possibilities: (*a*) the indicial equation has a double root; (*b*) the roots of the indicial equation differ by an integer. We now take a general look at these cases. For this purpose, let us consider the following differential equation which is highly important in mathematical physics:

$$x^2y'' + xg(x)y' + h(x)y = 0, \tag{2.27}$$

where the functions $g(x)$ and $h(x)$ are analytic at $x = 0$. Since the coefficients are not analytic at $x = 0$, the solution is of the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad (a_0 \neq 0). \tag{2.28}$$

We first expand $g(x)$ and $h(x)$ in power series,

$$g(x) = g_0 + g_1x + g_2x^2 + \cdots \quad h(x) = h_0 + h_1x + h_2x^2 + \cdots.$$

Then differentiating Eq. (2.28) term by term, we find

$$y'(x) = \sum_{m=0}^{\infty} (m + r)a_m x^{m+r-1}, \quad y''(x) = \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2}.$$

By inserting all these into Eq. (2.27) we obtain

$$x^r[r(r-1)a_0 + \cdots] + (g_0 + g_1x + \cdots)x^r(ra_0 + \cdots)$$
$$+ (h_0 + h_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0.$$

Equating the sum of the coefficients of each power of x to zero, as before, yields a system of equations involving the unknown coefficients a_m. The smallest power is x^r, and the corresponding equation is

$$[r(r-1) + g_0r + h_0]a_0 = 0.$$

Since by assumption $a_0 \neq 0$, we obtain

$$r(r-1) + g_0r + h_0 = 0 \quad \text{or} \quad r^2 + (g_0 - 1)r + h_0 = 0. \tag{2.29}$$

This is the indicial equation of the differential equation (2.27). We shall see that our series method will yield a fundamental system of solutions; one of the solutions will always be of the form (2.28), but for the form of other solution there will be three different possibilities corresponding to the following cases.

Case 1 The roots of the indicial equation are distinct and do not differ by an integer.
Case 2 The indicial equation has a double root.
Case 3 The roots of the indicial equation differ by an integer.

We now discuss these cases separately.

Case 1 Distinct roots not differing by an integer

This is the simplest case. Let r_1 and r_2 be the roots of the indicial equation (2.29). If we insert $r = r_1$ into the recurrence relation and determine the coefficients $a_1, a_2, \ldots$ successively, as before, then we obtain a solution

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots).$$

Similarly, by inserting the second root $r = r_2$ into the recurrence relation, we will obtain a second solution

$$y_2(x) = x^{r_2}(a_0^* + a_1^* x + a_2^* x^2 + \cdots).$$

Linear independence of y_1 and y_2 follows from the fact that y_1/y_2 is not constant because $r_1 - r_2$ is not an integer.

Case 2 Double roots

The indicial equation (2.29) has a double root r if, and only if, $(g_0 - 1)^2 - 4h_0 = 0$, and then $r = (1 - g_0)/2$. We may determine a first solution

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)\left(r = \frac{1-g_0}{2}\right) \tag{2.30}$$

as before. To find another solution we may apply the method of variation of parameters, that is, we replace constant c in the solution $cy_1(x)$ by a function $u(x)$ to be determined, such that

$$y_2(x) = u(x)y_1(x) \tag{2.31}$$

is a solution of Eq. (2.27). Inserting y_2 and the derivatives

$$y_2' = u'y_1 + uy_1' \qquad y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into the differential equation (2.27) we obtain

$$x^2(u''y_1 + 2u'y_1' + uy_1'') + xg(u'y_1 + uy_1') + huy_1 = 0$$

or

$$x^2 y_1 u'' + 2x^2 y_1' u' + xgy_1 u' + (x^2 y_1'' + xgy_1' + hy_1)u = 0.$$

Since y_1 is a solution of Eq. (2.27), the quantity inside the bracket vanishes; and the last equation reduces to

$$x^2 y_1 u'' + 2x^2 y_1' u' + xgy_1 u' = 0.$$

Dividing by $x^2 y_1$ and inserting the power series for g we obtain

$$u'' + \left(2\frac{y_1'}{y_1} + \frac{g_0}{x} + \cdots\right)u' = 0.$$

Here and in the following the dots designate terms which are constants or involve positive powers of x. Now from Eq. (2.30) it follows that

$$\frac{y_1'}{y_1} = \frac{x^{r-1}[ra_0 + (r+1)a_1 x + \cdots]}{x^r[a_0 + a_1 x + \cdots]} = \frac{1}{x}\,\frac{ra_0 + (r+1)a_1 x + \cdots}{a_0 + a_1 x + \cdots} = \frac{r}{x} + \cdots.$$

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Hence the last equation can be written

$$u'' + \left(\frac{2r + g_0}{x} + \cdots\right)u' = 0. \tag{2.32}$$

Since $r = (1 - g_0)/2$ the term $(2r + g_0)/x$ equals $1/x$, and by dividing by u' we thus have

$$\frac{u''}{u'} = -\frac{1}{x} + \cdots.$$

By integration we obtain

$$\ln u' = -\ln x + \cdots \quad \text{or} \quad u' = \frac{1}{x}e^{(\ldots)}.$$

Expanding the exponential function in powers of x and integrating once more, we see that the expression for u will be of the form

$$u = \ln x + k_1 x + k_2 x^2 + \cdots.$$

By inserting this into Eq. (2.31) we find that the second solution is of the form

$$y_2(x) = y_1(x)\ln x + x^r \sum_{m=1}^{\infty} A_m x^m. \tag{2.33}$$

Case 3 Roots differing by an integer

If the roots r_1 and r_2 of the indicial equation (2.29) differ by an integer, say, $r_1 = r$ and $r_2 = r - p$, where p is a positive integer, then we may always determine one solution as before, namely, the solution corresponding to r_1:

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots).$$

To determine a second solution y_2, we may proceed as in Case 2. The first steps are literally the same and yield Eq. (2.32). We determine $2r + g_0$ in Eq. (2.32). Then from the indicial equation (2.29), we find $-(r_1 + r_2) = g_0 - 1$. In our case, $r_1 = r$ and $r_2 = r - p$, therefore, $g_0 - 1 = p - 2r$. Hence in Eq. (2.32) we have $2r + g_0 = p + 1$, and we thus obtain

$$\frac{u''}{u'} = -\left(\frac{p+1}{x} + \cdots\right).$$

Integrating, we find

$$\ln u' = -(p+1)\ln x + \cdots \quad \text{or} \quad u' = x^{-(p+1)}e^{(\ldots)},$$

where the dots stand for some series of positive powers of x. By expanding the exponential function as before we obtain a series of the form

$$u' = \frac{1}{x^{p+1}} + \frac{k_1}{x^p} + \cdots + \frac{k_p}{x} + k_{p+1} + k_{p+2}x + \cdots.$$

Integrating, we have

$$u = -\frac{1}{px^p} - \cdots + k_p \ln x + k_{p+1}x + \cdots. \tag{2.34}$$

Multiplying this expression by the series

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

and remembering that $r_1 - p = r_2$ we see that $y_2 = uy_1$ is of the form

$$y_2(x) = k_p y_1(x) \ln x + x^{r_2} \sum_{m=0}^{\infty} a_m x^m. \tag{2.35}$$

While for a double root of Eq. (2.29) the second solution always contains a logarithmic term, the coefficient k_p may be zero and so the logarithmic term may be missing, as shown by the following example.

Example 2.15
Solve the differential equation

$$x^2 y'' + xy' + (x^2 - \tfrac{1}{4})y = 0.$$

Solution: Substituting Eq. (2.28) and its derivatives into this equation, we obtain

$$\sum_{m=0}^{\infty} [(m+r)(m+r-1) + (m+r) - \tfrac{1}{4}]a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0.$$

By equating the coefficient of x^r to zero we get the indicial equation

$$r(r-1) + r - \tfrac{1}{4} = 0 \quad \text{or} \quad r^2 = \tfrac{1}{4}.$$

The roots $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$ differ by an integer. By equating the sum of the coefficients of x^{s+r} to zero we find

$$[(r+1)r + (r-1) - \tfrac{1}{4}]a_1 = 0 \quad (s = 1). \tag{2.36a}$$

$$[(s+r)(s+r-1) + s + r - \tfrac{1}{4}]a_s + a_{s-2} = 0 \quad (s = 2, 3, \ldots). \tag{2.36b}$$

For $r = r_1 = \frac{1}{2}$, Eq. (2.36a) yields $a_1 = 0$, and the indicial equation (2.36b) becomes

$$(s+1)sa_s + a_{s-2} = 0.$$

From this and $a_1 = 0$ we obtain $a_3 = 0$, $a_5 = 0$, etc. Solving the indicial equation for a_s and setting $s = 2p$, we get

$$a_{2p} = -\frac{a_{2p-2}}{2p(2p+1)} \quad (p = 1, 2, \ldots).$$

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Hence the non-zero coefficients are

$$a_2 = -\frac{a_0}{3!}, \quad a_4 = -\frac{a_2}{4 \times 5} = \frac{a_0}{5!}, \quad a_6 = -\frac{a_0}{7!}, \text{etc.,}$$

and the solution y_1 is

$$y_1(x) = a_0\sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = a_0 \frac{\sin x}{\sqrt{x}}. \tag{2.37}$$

From Eq. (2.35) we see that a second independent solution is of the form

$$y_2(x) = k y_1(x) \ln x + x^{-1/2} \sum_{m=0}^{\infty} a_m x^m.$$

Substituting this and the derivatives into the differential equation, we see that the three expressions involving ln x and the expressions ky_1 and $-ky_1$ drop out. Simplifying the remaining equation, we thus obtain

$$2kxy_1' + \sum_{m=0}^{\infty} m(m-1) a_m x^{m-1/2} + \sum_{m=0}^{\infty} a_m x^{m+3/2} = 0.$$

From Eq. (2.37) we find $2kxy' = -ka_0 x^{1/2} + \cdots$. Since there is no further term involving $x^{1/2}$ and $a_0 \neq 0$, we must have $k = 0$. The sum of the coefficients of the power $x^{s-1/2}$ is

$$s(s-1)a_s + a_{s-2} \quad (s = 2, 3, \ldots).$$

Equating this to zero and solving for a_s, we have

$$a_s = -a_{s-2}/[s(s-1)] \quad (s = 2, 3, \ldots),$$

from which we obtain

$$a_2 = -\frac{a_0}{2!}, a_4 = -\frac{a_2}{4 \times 3} = \frac{a_0}{4!}, a_6 = -\frac{a_0}{6!}, \text{etc.,}$$

$$a_3 = -\frac{a_1}{3!}, a_5 = -\frac{a_3}{5 \times 4} = \frac{a_1}{5!}, a_7 = -\frac{a_1}{7!}, \text{etc.}$$

We may take $a_1 = 0$, because the odd powers would yield $a_1 y_1/a_0$. Then

$$y_2(x) = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = a_0 \frac{\cos x}{\sqrt{x}}.$$

Simultaneous equations

In some physics and engineering problems we may face simultaneous differential equations in two or more dependent variables. The general solution

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of simultaneous equations may be found by solving for each dependent variable separately, as shown by the following example

$$\left.\begin{aligned} Dx + 2y + 3x = 0 \\ 3x + Dy - 2y = 0 \end{aligned}\right\} \quad (D = d/dt)$$

which can be rewritten as

$$\left.\begin{aligned} (D+3)x + 2y = 0, \\ 3x + (D-2)y = 0. \end{aligned}\right\}$$

We then operate on the first equation with $(D - 2)$ and multiply the second by a factor 2:

$$\left.\begin{aligned} (D-2)(D+3)x + 2(D-2)y = 0, \\ 6x + 2(D-2)y = 0. \end{aligned}\right\}$$

Subtracting the first from the second leads to

$$(D^2 + D - 6)x - 6x = (D^2 + D - 12)x = 0,$$

which can easily be solved and its solution is of the form

$$x(t) = Ae^{3t} + Be^{-4t}.$$

Now inserting $x(t)$ back into the original equation to find y gives:

$$y(t) = -3Ae^{3t} + \tfrac{1}{2}Be^{-4t}.$$

The gamma and beta functions

The factorial notation $n! = n(n-1)(n-2)\cdots 3 \times 2 \times 1$ has proved useful in writing down the coefficients in some of the series solutions of the differential equations. However, this notation is meaningless when n is not a positive integer. A useful extension is provided by the gamma (or Euler) function, which is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx \quad (\alpha > 0) \tag{2.38}$$

and it follows immediately that

$$\Gamma(1) = \int_0^\infty e^{-x}dx = [-e^{-x}]_0^\infty = 1. \tag{2.39}$$

Integration by parts gives

$$\Gamma(\alpha+1) = \int_0^\infty e^{-x}x^{\alpha}dx = [-e^{-x}x^{\alpha}]_0^\infty + \alpha\int_0^\infty e^{-x}x^{\alpha-1}dx = \alpha\Gamma(\alpha). \tag{2.40}$$

When $\alpha = n$, a positive integer, repeated application of Eq. (2.40) and use of Eq. (2.39) gives

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n(n-1)\cdots 3 \times 2 \times \Gamma(1)$$
$$= n(n-1)\cdots 3 \times 2 \times 1 = n!.$$

Thus the gamma function is a generalization of the factorial function. Eq. (2.40) enables the values of the gamma function for any positive value of α to be calculated: thus

$$\Gamma(\tfrac{7}{2}) = (\tfrac{5}{2})\Gamma(\tfrac{5}{2}) = (\tfrac{5}{2})(\tfrac{3}{2})\Gamma(\tfrac{3}{2}) = (\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})\Gamma(\tfrac{1}{2}).$$

Write $u = +\sqrt{x}$ in Eq. (2.38) and we then obtain

$$\Gamma(\alpha) = 2\int_0^\infty u^{2\alpha-1}e^{-u^2}du,$$

so that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}.$$

The function $\Gamma(\alpha)$ has been tabulated for values of α between 0 and 1.

When $\alpha < 0$ we can define $\Gamma(\alpha)$ with the help of Eq. (2.40) and write

$$\Gamma(\alpha) = \Gamma(\alpha+1)/\alpha.$$

Thus

$$\Gamma(-\tfrac{3}{2}) = -\tfrac{2}{3}\Gamma(-\tfrac{1}{2}) = -\tfrac{2}{3}(-\tfrac{2}{1})\Gamma(\tfrac{1}{2}) = \tfrac{4}{3}\sqrt{\pi}.$$

When $\alpha \to 0$, $\int_0^\infty e^{-x}x^{\alpha-1}dx$ diverges so that $\Gamma(0)$ is not defined.

Another function which will be useful later is the beta function which is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \quad (p,q > 0). \tag{2.41}$$

Substituting $t = v/(1+v)$, this can be written in the alternative form

$$B(p,q) = \int_0^\infty v^{p-1}(1+v)^{-p-q}dv. \tag{2.42}$$

By writing $t' = 1-t$ we deduce that $B(p,q) = B(q,p)$.

The beta function can be expressed in terms of gamma functions as follows:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.43}$$

To prove this, write $x = at$ $(a > 0)$ in the integral (2.38) defining $\Gamma(\alpha)$, and it is straightforward to show that

$$\frac{\Gamma(\alpha)}{a^\alpha} = \int_0^\infty e^{-at}t^{\alpha-1}dt \tag{2.44}$$

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and, with $\alpha = p + q$, $a = 1 + v$, this can be written

$$\Gamma(p+q)(1+v)^{-p-q} = \int_0^\infty e^{-(1+v)t} t^{p+q-1} dt.$$

Multiplying by v^{p-1} and integrating with respect to v between 0 and ∞,

$$\Gamma(p+q)\int_0^\infty v^{p-1}(1+v)^{-p-q}dv = \int_0^\infty v^{p-1}dv \int_0^\infty e^{-(1+v)t} t^{p+q+1} dt.$$

Then interchanging the order of integration in the double integral on the right and using Eq. (2.42),

$$\begin{aligned}\Gamma(p+q)B(p,q) &= \int_0^\infty e^{-t} t^{p+q-1} dt \int_0^\infty e^{-vt} v^{p-1} dv \\ &= \int_0^\infty e^{-t} t^{p+q-1} \frac{\Gamma(p)}{t^p} dt, \quad \text{using Eq. (2.44)} \\ &= \Gamma(p)\int_0^\infty e^{-t} t^{q-1} dt = \Gamma(p)\Gamma(q).\end{aligned}$$

Example 2.15

Evaluate the integral $\int_0^\infty 3^{-4x^2} dx$.

Solution: We first notice that $3 = e^{\ln 3}$, so we can rewrite the integral as

$$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty (e^{\ln 3})^{(-4x^2)} dx = \int_0^\infty e^{-(4\ln 3)x^2} dx.$$

Now let $(4 \ln 3)x^2 = z$, then the integral becomes

$$\int_0^\infty e^{-z} d\left(\frac{z^{1/2}}{\sqrt{4\ln 3}}\right) = \frac{1}{2\sqrt{4\ln 3}}\int_0^\infty z^{-1/2}e^{-z}dz = \frac{\Gamma(\frac{1}{2})}{2\sqrt{4\ln 3}} = \frac{\sqrt{\pi}}{2\sqrt{4\ln 3}}.$$

Problems

2.1 Solve the following equations:
(*a*) $xdy/dx + y^2 = 1$;
(*b*) $dy/dx = (x+y)^2$.

2.2 Melting of a sphere of ice: Assume that a sphere of ice melts at a rate proportional to its surface area. Find an expression for the volume at any time t.

2.3 Show that $(3x^2 + y\cos x)dx + (\sin x - 4y^3)dy = 0$ is an exact differential equation and find its general solution.

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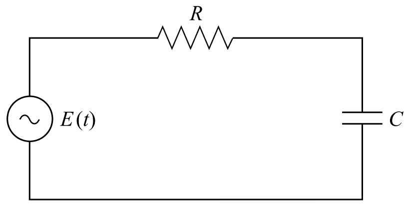

Figure 2.3. *RC* circuit.

2.4 *RC* circuits: A typical *RC* circuit is shown in Fig. 2.3. Find current flow $I(t)$ in the circuit, assuming $E(t) = E_0$.
Hint: the voltage drop across the capacitor is given Q/C, with $Q(t)$ the charge on the capacitor at time t.

2.5 Find a constant α such that $(x + y)^\alpha$ is an integrating factor of the equation

$$(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0.$$

What is the solution of this equation?

2.6 Solve $dy/dx + y = y^3 x$.

2.7 Solve:
(a) the equation $(D^2 - D - 12)y = 0$ with the boundary conditions $y = 0$, $Dy = 3$ when $t = 0$;
(b) the equation $(D^2 + 2D + 3)y = 0$ with the boundary conditions $y = 2$, $Dy = 0$ when $t = 0$;
(c) the equation $(D^2 - 2D + 1)y = 0$ with the boundary conditions $y = 5$, $Dy = 3$ when $t = 0$.

2.8 Find the particular integral of $(D^2 + 2D - 1)y = 3 + t^3$.

2.9 Find the particular integral of $(2D^2 + 5D + 7) = 3e^{2t}$.

2.10 Find the particular integral of $(3D^2 + D - 5)y = \cos 3t$.

2.11 Simple harmonic motion of a pendulum (Fig. 2.4): Suspend a ball of mass m at the end of a massless rod of length L and set it in motion swinging back and forth in a vertical plane. Show that the equation of motion of the ball is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$

where g is the local gravitational acceleration. Solve this pendulum equation for small displacements by replacing $\sin\theta$ by θ.

2.12 Forced oscillations with damping: If we allow an external driving force $F(t)$ in addition to damping (Example 2.12), the motion of the oscillator is governed by

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = F(t),$$

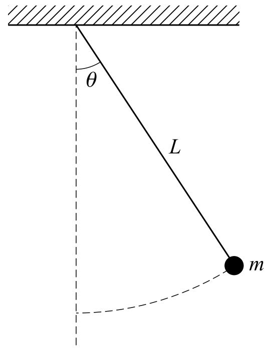

Figure 2.4. Simple pendulum.

a constant coefficient non-homogeneous equation. Solve this equation for $F(t) = A\cos(\omega t)$.

2.13 Solve the equation

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \quad (n \text{ constant}).$$

2.14 The first-order non-linear equation

$$\frac{dy}{dx} + y^2 + Q(x)y + R(x) = 0$$

is known as Riccati's equation. Show that, by use of a change of dependent variable

$$y = \frac{1}{z}\frac{dz}{dx};$$

Riccati's equation transforms into a second-order linear differential equation

$$\frac{d^2z}{dx^2} + Q(x)\frac{dz}{dx} + R(x)z = 0.$$

Sometimes Riccati's equation is written as

$$\frac{dy}{dx} + P(x)y^2 + Q(x)y + R(x) = 0.$$

Then the transformation becomes

$$y = -\frac{1}{P(x)}\frac{dz}{dx}$$

and the second-order equation takes the form

$$\frac{d^2z}{dx^2} + \left(Q + \frac{1}{P}\frac{dP}{dx}\right)\frac{dz}{dx} + PRz = 0.$$

2.15 Solve the equation $4x^2y'' + 4xy' + (x^2 - 1)y = 0$ by using Frobenius' method, where $y' = dy/dx$, and $y'' = d^2y/dx^2$.

2.16 Find a series solution, valid for large values of x, of the equation

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

2.17 Show that a series solution of Airy's equation $y'' - xy = 0$ is

$$y = a_0\left(1 + \frac{x^3}{2\times 3} + \frac{x^6}{2\times 3\times 5\times 6} + \cdots\right) + b_0\left(x + \frac{x^4}{3\times 4} + \frac{x^7}{3\times 4\times 6\times 7} + \cdots\right).$$

2.18 Show that Weber's equation $y'' + (n + \frac{1}{2} - \frac{1}{4}x^2)y = 0$ is reduced by the substitution $y = e^{-x^2/4}v$ to the equation $d^2v/dx^2 - x(dv/dx) + nv = 0$. Show that two solutions of this latter equation are

$$v_1 = 1 - \frac{n}{2!}x^2 + \frac{n(n-2)}{4!}x^4 - \frac{n(n-2)(n-4)}{6!}x^6 + - \cdots,$$

$$v_2 = x - \frac{(n-1)}{3!}x^3 + \frac{(n-1)(n-3)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)}{7!}x^7 + - \cdots.$$

2.19 Solve the following simultaneous equations

$$\left.\begin{aligned} Dx + y &= t^3 \\ Dy - x &= t \end{aligned}\right\} \quad (D = d/dt).$$

2.20 Evaluate the integrals:

(*a*) $\int_0^\infty x^3 e^{-x} dx.$

(*b*) $\int_0^\infty x^6 e^{-2x} dx$ (hint: let $y = 2x$).

(*c*) $\int_0^\infty \sqrt{y} e^{-y^2} dy$ (hint: let $y^2 = x$).

(*d*) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$ (hint : let $-\ln x = u$).

2.21 (*a*) Prove that $B(p, q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta d\theta.$

(*b*) Evaluate the integral $\int_0^1 x^4(1 - x)^3 dx.$

2.22 Show that $n! \approx \sqrt{2\pi n} n^n e^{-n}$. This is known as Stirling's factorial approximation or asymptotic formula for $n!$.