4

Fourier series and integrals

Fourier series are infinite series of sines and cosines which are capable of representing almost any periodic function whether continuous or not. Periodic functions that occur in physics and engineering problems are often very complicated and it is desirable to represent them in terms of simple periodic functions. Therefore the study of Fourier series is a matter of great practical importance for physicists and engineers.

The first part of this chapter deals with Fourier series. Basic concepts, facts, and techniques in connection with Fourier series will be introduced and developed, along with illustrative examples. They are followed by Fourier integrals and Fourier transforms.

Periodic functions

If function $f(x)$ is defined for all x and there is some positive constant P such that

$$f(x+P) = f(x) \tag{4.1}$$

then we say that $f(x)$ is periodic with a period P (Fig. 4.1). From Eq. (4.1) we also

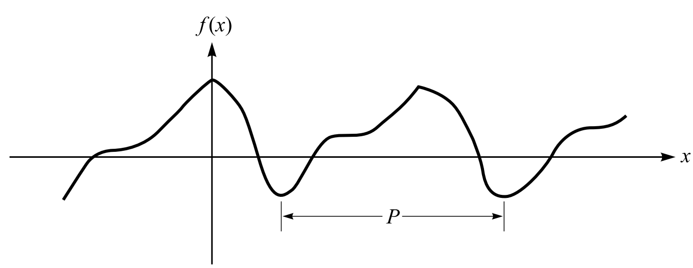

Figure 4.1. A general periodic function.

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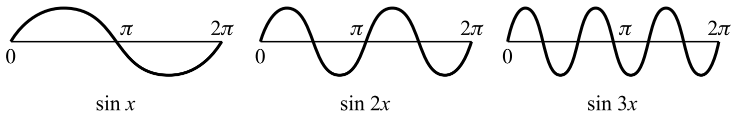

Figure 4.2. Sine functions.

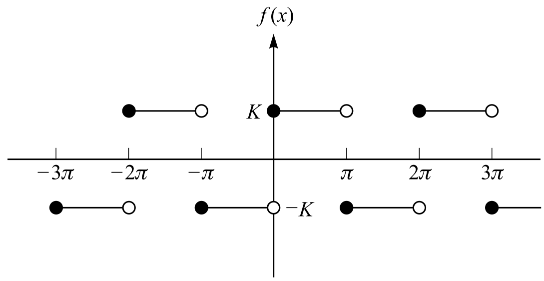

Figure 4.3. A square wave function.

have, for all x and any integer n,

$$f(x + nP) = f(x).$$

That is, every periodic function has arbitrarily large periods and contains arbitrarily large numbers in its domain. We call P the fundamental (or least) period, or simply the period.

A periodic function need not be defined for all values of its independent variable. For example, tan x is undefined for the values $x = (\pi/2) + n\pi$. But tan x is a periodic function in its domain of definition, with π as its fundamental period: $\tan(x + \pi) = \tan x$.

Example 4.1

(*a*) The period of $\sin x$ is 2π, since $\sin(x + 2\pi)$, $\sin(x + 4\pi)$, $\sin(x + 6\pi), \ldots$ are all equal to sin x, but 2π is the least value of P. And, as shown in Fig. 4.2, the period of sin nx is $2\pi/n$, where n is a positive integer.

(*b*) A constant function has any positive number as a period. Since $f(x) = c$ (const.) is defined for all real x, then, for every positive number P, $f(x + P) = c = f(x)$. Hence P is a period of f. Furthermore, f has no fundamental period.

(*c*)

$$f(x) = \begin{cases} K & \text{for} \quad 2n\pi x \le (2n + 1)\pi \\ -K & \text{for} \quad (2n + 1)\pi \le x < (2n + 2)\pi \end{cases} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

is periodic of period 2π (Fig. 4.3).

Fourier series; Euler–Fourier formulas

If the general periodic function $f(x)$ is defined in an interval $-\pi \le x \le \pi$, the Fourier series of $f(x)$ in $[-\pi, \pi]$ is defined to be a trigonometric series of the form

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots, \tag{4.2}$$

where the numbers $a_0, a_1, a_2, \ldots, b_1, b_2, b_3, \ldots$ are called the Fourier coefficients of $f(x)$ in $[-\pi, \pi]$. If this expansion is possible, then our power to solve physical problems is greatly increased, since the sine and cosine terms in the series can be handled individually without difficulty. Joseph Fourier (1768–1830), a French mathematician, undertook the systematic study of such expansions. In 1807 he submitted a paper (on heat conduction) to the Academy of Sciences in Paris and claimed that every function defined on the closed interval $[-\pi, \pi]$ could be represented in the form of a series given by Eq. (4.2); he also provided integral formulas for the coefficients a_n and b_n. These integral formulas had been obtained earlier by Clairaut in 1757 and by Euler in 1777. However, Fourier opened a new avenue by claiming that these integral formulas are well defined even for very arbitrary functions and that the resulting coefficients are identical for different functions that are defined within the interval. Fourier's paper was rejected by the Academy on the grounds that it lacked mathematical rigor, because he did not examine the question of the convergence of the series.

The trigonometric series (4.2) is the only series which corresponds to $f(x)$. Questions concerning its convergence and, if it does, the conditions under which it converges to $f(x)$ are many and difficult. These problems were partially answered by Peter Gustave Lejeune Dirichlet (German mathematician, 1805–1859) and will be discussed briefly later.

Now let us assume that the series exists, converges, and may be integrated term by term. Multiplying both sides by cos mx, then integrating the result from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos mx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\, dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\, dx. \tag{4.3}$$

Now, using the following important properties of sines and cosines:

$$\int_{-\pi}^{\pi} \cos mx\, dx = \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{if } m = 1, 2, 3, \ldots,$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } n \ne m, \\ \pi & \text{if } n = m, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0, \quad \text{for all } m, n > 0,$$

we find that all terms on the right hand side of Eq. (4.3) except one vanish:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = \text{integers}; \tag{4.4a}$$

the expression for a_0 can be obtained from the general expression for a_n by setting $n = 0$.

Similarly, if Eq. (4.2) is multiplied through by sin mx and the result is integrated from $-\pi$ to π, all terms vanish save that involving the square of sin nx, and so we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \tag{4.4b}$$

Eqs. (4.4a) and (4.4b) are known as the Euler–Fourier formulas.

From the definition of a definite integral it follows that, if $f(x)$ is single-valued and continuous within the interval $[-\pi, \pi]$ or merely piecewise continuous (continuous except at a finite numbers of finite jumps in the interval), the integrals in Eqs. (4.4) exist and we may compute the Fourier coefficients of $f(x)$ by Eqs. (4.4). If there exists a finite discontinuity in $f(x)$ at the point x_0 (Fig. 4.1), the coefficients a_0, a_n, b_n are determined by integrating first to $x = x_0$ and then from x_0 to π, as

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{x_0} f(x) \cos nx \, dx + \int_{x_0}^{\pi} f(x) \cos nx \, dx \right], \tag{4.5a}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{x_0} f(x) \sin nx \, dx + \int_{x_0}^{\pi} f(x) \sin nx \, dx \right]. \tag{4.5b}$$

This procedure may be extended to any finite number of discontinuities.

Example 4.2

Find the Fourier series which represents the function

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ +k & 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x),$$

in the interval $-\pi \le x \le \pi$.

Solution: The Fourier coefficients are readily calculated:

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$
$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$
$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right] = \frac{2k}{n\pi}(1-\cos n\pi)$$

Now cos $n\pi = -1$ for odd n, and cos $n\pi = 1$ for even n. Thus

$$b_1 = 4k/\pi, \quad b_2 = 0, \quad b_3 = 4k/3\pi, \quad b_4 = 0, \quad b_5 = 4k/5\pi, \ldots$$

and the corresponding. Fourier series is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

For the special case $k = \pi/2$, the Fourier series becomes

$$2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots.$$

The first two terms are shown in Fig. 4.4, the solid curve is their sum. We will see that as more and more terms in the Fourier series expansion are included, the sum more and more nearly approaches the shape of $f(x)$. This will be further demonstrated by next example.

Example 4.3
Find the Fourier series that represents the function defined by

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin t, & 0 < t < \pi \end{cases} \quad \text{in the interval } -\pi < t < \pi.$$

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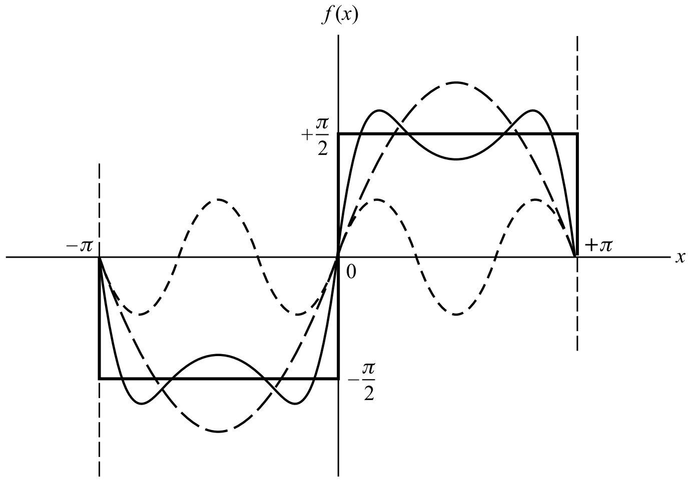

Figure 4.4. The first two partial sums.

Solution:

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \cos nt\,dt + \int_{0}^{\pi} \sin t \cos nt\,dt\right]$$

$$= -\frac{1}{2\pi}\left[\frac{\cos(1-n)t}{1-n} + \frac{\cos(1+n)t}{1+n}\right]\Bigg|_0^{\pi} = \frac{\cos n\pi + 1}{\pi(1-n)^2}, \qquad n \neq 1,$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin t \cos t\,dt = \frac{1}{\pi}\frac{\sin^2 t}{2}\Bigg|_0^{\pi} = 0,$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \sin nt\,dt + \int_{0}^{\pi} \sin t \sin nt\,dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n}\right]_0^{\pi} = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t\,dt = \frac{1}{\pi}\left[\frac{t}{2} - \frac{\sin\,2t}{4}\right]_0^{\pi} = \frac{1}{2}.$$

Accordingly the Fourier expansion of $f(t)$ in $[-\pi, \pi]$ may be written

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi}\left[\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right].$$

The first three partial sums $S_n (n = 1, 2, 3)$ are shown in Fig. 4.5: $S_1 = 1/\pi$, $S_2 = 1/\pi + \sin t/2$, and $S_3 = 1/\pi + \sin(t)/2 - 2\cos(2t)/3$.

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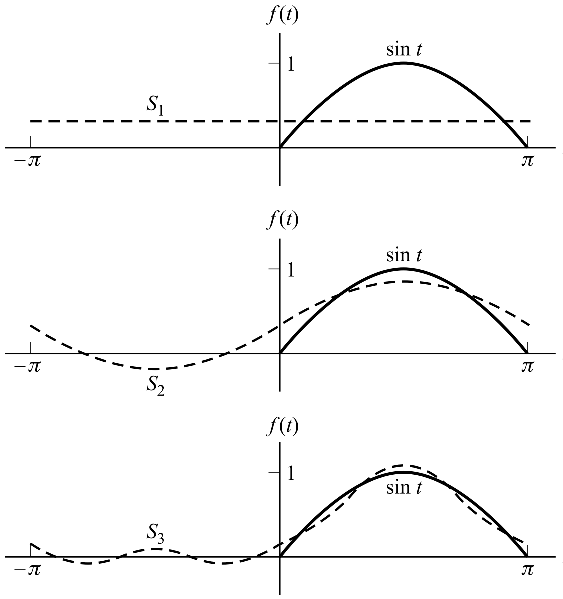

Figure 4.5. The first three partial sums of the series.

Gibb's phenomena

From Figs. 4.4 and 4.5, two features of the Fourier expansion should be noted:

(a) at the points of the discontinuity, the series yields the mean value;

(b) in the region immediately adjacent to the points of discontinuity, the expansion overshoots the original function. This effect is known as the *Gibb's phenomena* and occurs in all order of approximation.

Convergence of Fourier series and Dirichlet conditions

The serious question of the convergence of Fourier series still remains: if we determine the Fourier coefficients a_n, b_n of a given function $f(x)$ from Eq. (4.4) and form the Fourier series given on the right hand side of Eq. (4.2), will it converge toward $f(x)$? This question was partially answered by Dirichlet. Here is a restatement of the results of his study, which is often called Dirichlet's theorem:

(1) If $f(x)$ is defined and single-valued except at a finite number of point in $[-\pi, \pi]$,

(2) if $f(x)$ is periodic outside $[-\pi, \pi]$ with period 2π (that is, $f(x+2\pi) = f(x)$), and

(3) if $f(x)$ and $f'(x)$ are piecewise continuous in $[-\pi, \pi]$,

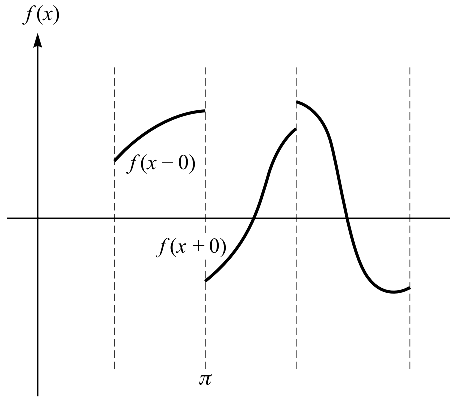

Figure 4.6. A piecewise continuous function.

then the series on the right hand side of Eq. (4.2), with coefficients a_n and b_n given by Eqs. (4.4), converges to

(*i*) $f(x)$, if x is a point of continuity, or
(*ii*) $\frac{1}{2}[f(x+0)+f(x-0)]$, if x is a point of discontinuity as shown in Fig. 4.6,

where $f(x+0)$ and $f(x-0)$ are the right and left hand limits of $f(x)$ at x and represent $\lim_{\varepsilon\to 0} f(x+\varepsilon)$ and $\lim_{\varepsilon\to 0} f(x-\varepsilon)$ respectively, where $\varepsilon > 0$.

The proof of Dirichlet's theorem is quite technical and is omitted in this treatment. The reader should remember that the Dirichlet conditions (1), (2), and (3) imposed on $f(x)$ are sufficient but not necessary. That is, if the above conditions are satisfied the convergence is guaranteed; but if they are not satisfied, the series may or may not converge. The Dirichlet conditions are generally satisfied in practice.

Half-range Fourier series

Unnecessary work in determining Fourier coefficients of a function can be avoided if the function is odd or even. A function $f(x)$ is called odd if $f(-x) = -f(x)$ and even if $f(x)$ $f(-x) = f(x)$. It is easy to show that in the Fourier series corresponding to an odd function $f_o(x)$, only sine terms can be present in the series expansion in the interval $-\pi < x < \pi$, for

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f_o(x)\cos nx\,dx + \int_{0}^{\pi} f_o(x)\cos nx\,dx\right]$$
$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} f_o(x)\cos nx\,dx + \int_{0}^{\pi} f_o(x)\cos nx\,dx\right] = 0 \qquad n = 0, 1, 2, \ldots, \quad (4.6a)$$

but

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f_o(x)\sin nx\,dx + \int_0^{\pi} f_o(x)\sin nx\,dx\right]$$
$$= \frac{2}{\pi}\int_0^{\pi} f_o(x)\sin nx\,dx \qquad n = 1, 2, 3, \ldots. \tag{4.6b}$$

Here we have made use of the fact that $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$. Accordingly, the Fourier series becomes

$$f_o(x) = b_1 \sin x + b_2 \sin 2x + \cdots.$$

Similarly, in the Fourier series corresponding to an even function $f_e(x)$, only cosine terms (and possibly a constant) can be present. Because in this case, $f_e(x)\sin nx$ is an odd function and accordingly $b_n = 0$ and the a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos nx\,dx \qquad n = 0, 1, 2, \ldots. \tag{4.7}$$

Note that the Fourier coefficients a_n and b_n, Eqs. (4.6) and (4.7) are computed in the interval $(0, \pi)$ which is *half* of the interval $(-\pi, \pi)$. Thus, the Fourier sine or cosine series in this case is often called a half-range Fourier series.

Any arbitrary function (neither even nor odd) can be expressed as a combination of $f_e(x)$ and $f_o(x)$ as

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] = f_e(x) + f_o(x).$$

When a half-range series corresponding to a given function is desired, the function is generally defined in the interval $(0, \pi)$ and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval $(-\pi, 0)$.

Change of interval

A Fourier expansion is not restricted to such intervals as $-\pi < x < \pi$ and $0 < x < \pi$. In many problems the period of the function to be expanded may be some other interval, say $2L$. How then can the Fourier series developed above be applied to the representation of periodic functions of arbitrary period? The problem is not a difficult one, for basically all that is involved is to change the variable. Let

$$z = \frac{\pi}{L}x \tag{4.8a}$$

then

$$f(z) = f(\pi x/L) = F(x). \tag{4.8b}$$

Thus, if $f(z)$ is expanded in the interval $-\pi < z < \pi$, the coefficients being determined by expressions of the form of Eqs. (4.4a) and (4.4b), the coefficients for the

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expansion of $F(x)$ in the interval $-L < x < L$ may be obtained merely by substituting Eqs. (4.8) into these expressions. We have then

$$a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi}{L}x\,dx \qquad n = 0, 1, 2, 3, \ldots, \tag{4.9a}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi}{L}x\,dx, \qquad n = 1, 2, 3, \ldots. \tag{4.9b}$$

The possibility of having expanding functions in which the period is other than 2π increases the usefulness of Fourier expansion. As an example, consider the value of L, it is obvious that the larger the value of L, the larger the basic period of the function being expanded. As $L \to \infty$, the function would not be periodic at all. We will see later that in such cases the Fourier series becomes a Fourier integral.

Parseval's identity

Parseval's identity states that:

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2), \tag{4.10}$$

if a_n and b_n are coefficients of the Fourier series of $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

It is easy to prove this identity. Assuming that the Fourier series corresponding to $f(x)$ converges to $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right).$$

Multiplying by $f(x)$ and integrating term by term from $-L$ to L, we obtain

$$\begin{aligned}\int_{-L}^{L}[f(x)]^2 dx &= \frac{a_0}{2}\int_{-L}^{L} f(x)dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\right\} \\ &= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right),\end{aligned} \tag{4.11}$$

where we have used the results

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = La_n, \qquad \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = Lb_n, \qquad \int_{-L}^{L} f(x)dx = La_0.$$

The required result follows on dividing both sides of Eq. (4.11) by L.

Parseval's identity shows a relation between the average of the square of $f(x)$ and the coefficients in the Fourier series for $f(x)$:

the average of $\{f(x)\}^2$ is $\int_{-L}^{L}[f(x)]^2 dx/2L$;
the average of $(a_0/2)$ is $(a_0/2)^2$;
the average of $(a_n \cos nx)$ is $a_n^2/2$;
the average of $(b_n \sin nx)$ is $b_n^2/2$.

Example 4.4
Expand $f(x) = x, 0 < x < 2$, in a half-range cosine series, then write Parseval's identity corresponding to this Fourier cosine series.

Solution: We first extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 4.7. Then $2L = 4, L = 2$. Thus $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx$$
$$= \left[x\cdot\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 1\cdot\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$
$$= \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{if } n \neq 0.$$

If $n = 0$,

$$a_0 = \int_0^L x dx = 2.$$

Then

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}.$$

We now write Parseval's identity. We first compute the average of $[f(x)]^2$:

$$\text{the average of } [f(x)]^2 = \tfrac{1}{2}\int_{-2}^{2}\{f(x)\}^2 dx = \tfrac{1}{2}\int_{-2}^{2}x^2 dx = \frac{8}{3},$$

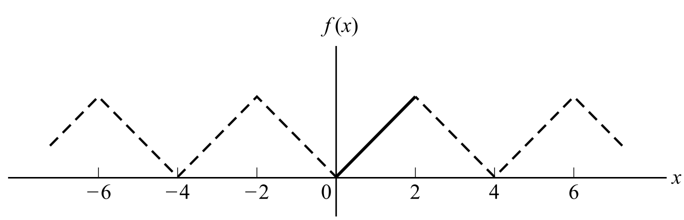

Figure 4.7.

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then the average

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{(2)^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4} (\cos n\pi - 1)^2.$$

Parseval's identity now becomes

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right),$$

or

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

which shows that we can use Parseval's identity to find the sum of an infinite series. With the help of the above result, we can find the sum S of the following series:

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots + \frac{1}{n^4} + \cdots.$$

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right)$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$= \frac{\pi^4}{96} + \frac{S}{16}$$

from which we find $S = \pi^4/90$.

Alternative forms of Fourier series

Up to this point the Fourier series of a function has been written as an infinite series of sines and cosines, Eq. (4.2):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

This can be converted into other forms. In this section, we just discuss two alternative forms. Let us first write, with $\pi/L = \alpha$

$$a_n \cos n\alpha x + b_n \sin n\alpha x = \sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\alpha x + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\alpha x\right).$$

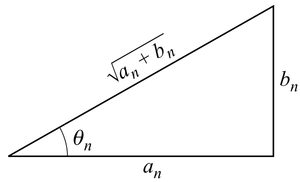

Figure 4.8.

Now let (see Fig. 4.8)

$$\cos\theta_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \sin\theta_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}, \quad \text{so} \quad \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right),$$

$$C_n = \sqrt{a_n^2 + b_n^2}, \qquad C_0 = \tfrac{1}{2}a_0,$$

then we have the trigonometric identity

$$a_n \cos n\alpha x + b_n \sin n\alpha x = C_n \cos(n\alpha x - \theta_n),$$

and accordingly the Fourier series becomes

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\alpha x - \theta_n). \tag{4.12}$$

In this new form, the Fourier series represents a periodic function as a sum of sinusoidal components having different frequencies. The sinusoidal component of frequency $n\alpha$ is called the nth harmonic of the periodic function. The first harmonic is commonly called the fundamental component. The angles θ_n and the coefficients C_n are known as the phase angle and amplitude.

Using Euler's identities $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ where $i^2 = -1$, the Fourier series for $f(x)$ can be converted into complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \tag{4.13a}$$

where

$$\begin{aligned} c_{\pm n} &= a_n \mp i b_n \\ &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx, \quad \text{for } n > 0. \end{aligned} \tag{4.13b}$$

Eq. (4.13a) is obtained on the understanding that the Dirichlet conditions are satisfied and that $f(x)$ is continuous at x. If $f(x)$ is discontinuous at x, the left hand side of Eq. (4.13a) should be replaced by $[f(x+0) + f(x-0)]/2$.

The exponential form (4.13a) can be considered as a basic form in its own right: it is not obtained by transformation from the trigonometric form, rather it is constructed directly from the given function. Furthermore, in the complex representation defined by Eqs. (4.13a) and (4.13b), a certain symmetry between the expressions for a function and for its Fourier coefficients is evident. In fact the expressions (4.13a) and (4.13b) are of essentially the same structure, as the following correlation reveals:

$$x \sim L,\ f(x) \sim c_n \equiv c(n), \quad e^{in\pi x/L} \sim e^{-in\pi x/L}, \sum_{n=-\infty}^{\infty}(\ ) \sim \frac{1}{2L}\int_{-L}^{L}(\ )dx.$$

This duality is worthy of note, and as our development proceeds to the Fourier integral, it will become more striking and fundamental.

Integration and differentiation of a Fourier series

The Fourier series of a function $f(x)$ may always be integrated term-by-term to give a new series which converges to the integral of $f(x)$. If $f(x)$ is a continuous function of x for all x, and is periodic (of period 2π) outside the interval $-\pi < x < \pi$, then term-by-term differentiation of the Fourier series of $f(x)$ leads to the Fourier series of $f'(x)$, provided $f'(x)$ satisfies Dirichlet's conditions.

Vibrating strings

The equation of motion of transverse vibration

There are numerous applications of Fourier series to solutions of boundary value problems. Here we consider one of them, namely vibrating strings. Let a string of length L be held fixed between two points (0, 0) and (L, 0) on the x-axis, and then given a transverse displacement parallel to the y-axis. Its subsequent motion, with no external forces acting on it, is to be considered; this is described by finding the displacement y as a function of x and t (if we consider only vibration in one plane, and take the xy plane as the plane of vibration). We will assume that ρ, the mass per unit length is uniform over the entire length of the string, and that the string is perfectly flexible, so that it can transmit tension but not bending or shearing forces.

As the string is drawn aside from its position of rest along the x-axis, the resulting increase in length causes an increase in tension, denoted by P. This tension at any point along the string is always in the direction of the tangent to the string at that point. As shown in Fig. 4.9, a force $P(x)A$ acts at the left hand side of an element ds, and a force $P(x+dx)A$ acts at the right hand side, where A is the cross-sectional area of the string. If α is the inclination to the horizontal, then

$$F_x \cong AP\cos(\alpha + d\alpha) - AP\cos\alpha, \qquad F_y \cong AP\sin(\alpha + d\alpha) - AP\sin\alpha.$$

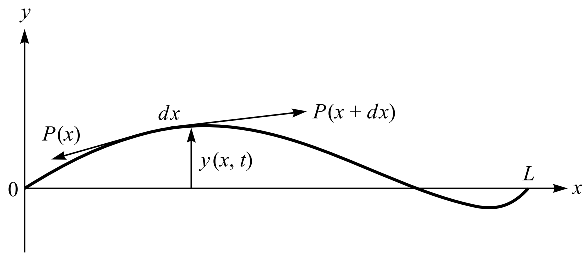

Figure 4.9. A vibrating string.

We limit the displacement to small values, so that we may set

$$\cos\alpha = 1 - \alpha^2/2, \qquad \sin\alpha \cong \alpha \cong \tan\alpha = dy/dx,$$

then

$$F_y = AP\left[\left(\frac{dy}{dx}\right)_{x+dx} - \left(\frac{dy}{dx}\right)_x\right] = AP\frac{d^2y}{dx^2}dx.$$

Using Newton's second law, the equation of motion of transverse vibration of the element becomes

$$\rho Adx\frac{\partial^2 y}{\partial t^2} = AP\frac{\partial^2 y}{\partial x^2}dx, \qquad \text{or} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \quad v = \sqrt{P/\rho}.$$

Thus the transverse displacement of the string satisfies the partial differential wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \tag{4.14}$$

with the following boundary conditions: $y(0,t) = y(L,t) = 0$, $\partial y/\partial t = 0$, $y(x,0) = f(x)$; where $f(x)$ describes the initial shape (position) of the string, and v is the velocity of propagation of the wave along the string.

Solution of the wave equation

To solve this boundary value problem, let us try the method of separation variables:

$$y(x,t) = X(x)T(t). \tag{4.15}$$

Substituting this into Eq. (4.14) yields

$$(1/X)(d^2X/dx^2) = (1/v^2T)(d^2T/dt^2).$$

Since the left hand side is a function of x only and the right hand side is a function of time only, they must be equal to a common separation constant, which we will call $-\lambda^2$. Then we have

$$d^2X/dx^2 = -\lambda^2 X, \quad X(0) = X(L) = 0 \tag{4.16a}$$

and

$$d^2T/dt^2 = -\lambda^2 v^2 T \quad dT/dt = 0 \quad \text{at } t = 0. \tag{4.16b}$$

Both of these equations are typical eigenvalue problems: we have a differential equation containing a parameter λ, and we seek solutions satisfying certain boundary conditions. If there are special values of λ for which non-trivial solutions exist, we call these eigenvalues, and the corresponding solutions eigensolutions or eigenfunctions.

The general solution of Eq. (4.16a) can be written as

$$X(x) = A_1 \sin(\lambda x) + B_1 \cos(\lambda x).$$

Applying the boundary conditions

$$X(0) = 0 \Rightarrow B_1 = 0,$$

and

$$X(L) = 0 \Rightarrow A_1 \sin(\lambda L) = 0$$

$A_1 = 0$ is the trivial solution $X = 0$ (so $y = 0$); hence we must have $\sin(\lambda L) = 0$, that is,

$$\lambda L = n\pi, \quad n = 1, 2, \ldots,$$

and we obtain a series of eigenvalues

$$\lambda_n = n\pi/L, \quad n = 1, 2, \ldots$$

and the corresponding eigenfunctions

$$X_n(x) = \sin(n\pi/L)x, \quad n = 1, 2, \ldots .$$

To solve Eq. (4.16b) for $T(t)$ we must use one of the values λ_n found above. The general solution is of the form

$$T(t) = A_2 \cos(\lambda_n vt) + B_2 \sin(\lambda_n vt).$$

The boundary condition leads to $B_2 = 0$.

The general solution of Eq. (4.14) is hence a linear superposition of the solutions of the form

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) \cos(n\pi vt/L); \tag{4.17}$$

the A_n are as yet undetermined constants. To find A_n, we use the boundary condition $y(x, t) = f(x)$ at $t = 0$, so that Eq. (4.17) reduces to

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L).$$

Do you recognize the infinite series on the right hand side? It is a Fourier sine series. To find A_n, multiply both sides by $\sin(m\pi x/L)$ and then integrate with respect to x from 0 to L and we obtain

$$A_m = \frac{2}{L}\int_0^L f(x)\sin(m\pi x/L)dx, \qquad m = 1, 2, \ldots$$

where we have used the relation

$$\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)dx = \frac{L}{2}\delta_{mn}.$$

Eq. (4.17) now gives

$$y(x, t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\right]\sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}. \tag{4.18}$$

The terms in this series represent the natural modes of vibration. The frequency of the nth normal mode f_n is obtained from the term involving $\cos(n\pi vt/L)$ and is given by

$$2\pi f_n = n\pi v/L \quad \text{or} \quad f_n = nv/2L.$$

All frequencies are integer multiples of the lowest frequency f_1. We call f_1 the fundamental frequency or first harmonic, and f_2 and f_3 the second and third harmonics (or first and second overtones) and so on.

RLC circuit

Another good example of application of Fourier series is an *RLC* circuit driven by a variable voltage $E(t)$ which is periodic but not necessarily sinusoidal (see Fig. 4.10). We want to find the current $I(t)$ flowing in the circuit at time t.

According to Kirchhoff's second law for circuits, the impressed voltage $E(t)$ equals the sum of the voltage drops across the circuit components. That is,

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t),$$

where Q is the total charge in the capacitor C. But $I = dQ/dt$, thus differentiating the above differential equation once we obtain

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}.$$

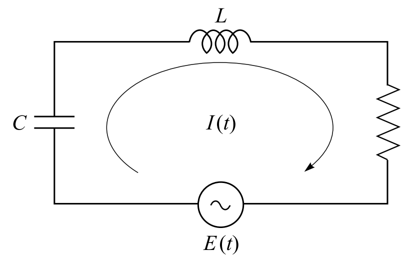

Figure 4.10. The *RLC* circuit.

Under steady-state conditions the current $I(t)$ is also periodic, with the same period P as for $E(t)$. Let us assume that both $E(t)$ and $I(t)$ possess Fourier expansions and let us write them in their complex forms:

$$E(t) = \sum_{n=-\infty}^{\infty} E_n e^{in\omega t}, \qquad I(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \qquad (\omega = 2\pi/P).$$

Furthermore, we assume that the series can be differentiated term by term. Thus

$$\frac{dE}{dt} = \sum_{n=-\infty}^{\infty} in\omega E_n e^{in\omega t}, \quad \frac{dI}{dt} = \sum_{n=-\infty}^{\infty} in\omega c_n e^{in\omega t}, \quad \frac{d^2 I}{dt^2} = \sum_{n=-\infty}^{\infty} (-n^2\omega^2) c_n e^{in\omega t}.$$

Substituting these into the last (second-order) differential equation and equating the coefficients with the same exponential $e^{in\alpha t}$, we obtain

$$\left(-n^2\omega^2 L + in\omega R + 1/C\right) c_n = in\omega E_n.$$

Solving for c_n

$$c_n = \frac{in\omega/L}{[(1/CL)^2 - n^2\omega^2] + i(R/L)n\omega} E_n.$$

Note that $1/LC$ is the natural frequency of the circuit and R/L is the attenuation factor of the circuit. The Fourier coefficients for $E(t)$ are given by

$$E_n = \frac{1}{P}\int_{-P/2}^{P/2} E(t) e^{-in\omega t} dt.$$

The current $I(t)$ in the circuit is given by

$$I(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}.$$

Orthogonal functions

Many of the properties of Fourier series considered above depend on orthogonal properties of sine and cosine functions

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0, \qquad \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \quad (m \neq n).$$

In this section we seek to generalize this orthogonal property. To do so we first recall some elementary properties of real vectors in three-dimensional space.

Two vectors **A** and **B** are called orthogonal if $\mathbf{A} \cdot \mathbf{B} = 0$. Although not geometrically or physically obvious, we generalize these ideas to think of a function, say $A(x)$, as being an infinite-dimensional vector (a vector with an infinity of components), the value of each component being specified by substituting a particular value of x taken from some interval (a, b), and two functions, $A(x)$ and $B(x)$ are orthogonal in (a, b) if

$$\int_a^b A(x)B(x)dx = 0. \tag{4.19}$$

The left-side of Eq. (4.19) is called the scalar product of $A(x)$ and $B(x)$ and denoted by, in the Dirac bracket notation, $\langle A(x)|B(x)\rangle$. The first factor in the bracket notation is referred to as the bra and the second factor as the ket, so together they comprise the bracket.

A vector **A** is called a unit vector or normalized vector if its magnitude is unity: $\mathbf{A} \cdot \mathbf{A} = A^2 = 1$. Extending this concept, we say that the function $A(x)$ is normal or normalized in (a, b) if

$$\langle A(x)|A(x)\rangle = \int_a^b A(x)A(x)dx = 1. \tag{4.20}$$

If we have a set of functions $\varphi_i(x), i = 1, 2, 3, \ldots,$ having the properties

$$\langle \varphi_m(x)|\varphi_n(x)\rangle = \int_a^b \varphi_m(x)\varphi_n(x)dx = \delta_{mn}, \tag{4.20a}$$

where δ_{nm} is the Kronecker delta symbol, we then call such a set of functions an orthonormal set in (a, b). For example, the set of functions $\varphi_m(x) = (2/\pi)^{1/2}\sin(mx), m = 1, 2, 3, \ldots$ is an orthonormal set in the interval $0 \le x \le \pi$.

Just as in three-dimensional vector space, any vector **A** can be expanded in the form $\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$, we can consider a set of orthonormal functions φ_i as base vectors and expand a function $f(x)$ in terms of them, that is,

$$f(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x) \qquad a \le x \le b; \tag{4.21}$$

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the series on the right hand side is called an orthonormal series; such series are generalizations of Fourier series. Assuming that the series on the right converges to $f(x)$, we can then multiply both sides by $\varphi_m(x)$ and integrate both sides from a to b to obtain

$$c_m = \langle f(x)|\varphi_m(x)\rangle = \int_a^b f(x)\varphi_m(x)dx; \tag{4.21a}$$

c_m can be called the generalized Fourier coefficients.

Multiple Fourier series

A Fourier expansion of a function of two or three variables is often very useful in many applications. Let us consider the case of a function of two variables, say $f(x, y)$. For example, we can expand $f(x, y)$ into a double Fourier sine series

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}, \tag{4.22}$$

where

$$B_{mn} = \frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} f(x, y) \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} dxdy. \tag{4.22a}$$

Similar expansions can be made for cosine series and for series having both sines and cosines.

To obtain the coefficients B_{mn}, let us rewrite $f(x, y)$ as

$$f(x, y) = \sum_{m=1}^{\infty} C_m \sin\frac{m\pi x}{L_1}, \tag{4.23}$$

where

$$C_m = \sum_{n=1}^{\infty} B_{mn} \sin\frac{n\pi y}{L_2}. \tag{4.23a}$$

Now we can consider Eq. (4.23) as a Fourier series in which y is kept constant so that the Fourier coefficients C_m are given by

$$C_m = \frac{2}{L_1}\int_0^{L_1} f(x, y) \sin\frac{m\pi x}{L_1} dx. \tag{4.24}$$

On noting that C_m is a function of y, we see that Eq. (4.23a) can be considered as a Fourier series for which the coefficients B_{mn} are given by

$$B_{mn} = \frac{2}{L_2}\int_0^{L_2} C_m \sin\frac{n\pi y}{L_2} dy.$$

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Substituting Eq. (4.24) for C_m into the above equation, we see that B_{mn} is given by Eq. (4.22a).

Similar results can be obtained for cosine series or for series containing both sines and cosines. Furthermore, these ideas can be generalized to triple Fourier series, etc. They are very useful in solving, for example, wave propagation and heat conduction problems in two or three dimensions. Because they lie outside of the scope of this book, we have to omit these interesting applications.

Fourier integrals and Fourier transforms

The properties of Fourier series that we have thus far developed are adequate for handling the expansion of any periodic function that satisfies the Dirichlet conditions. But many problems in physics and engineering do not involve periodic functions, and it is therefore desirable to generalize the Fourier series method to include non-periodic functions. A non-periodic function can be considered as a limit of a given periodic function whose period becomes infinite, as shown in Examples 4.5 and 4.6.

Example 4.5

Consider the periodic functions $f_L(x)$

$$f_L(x) = \begin{cases} 0 \text{ when} & -L/2 < x < -1 \\ 1 \text{ when} & -1 < x < 1, \\ 0 \text{ when} & 1 < x < L/2 \end{cases}$$

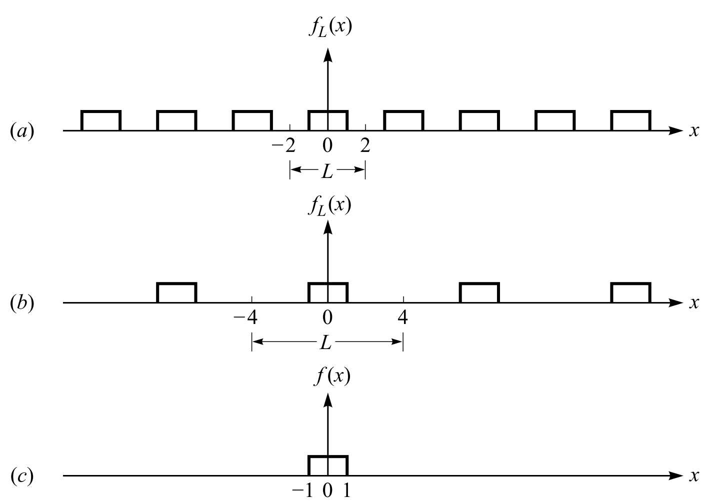

Figure 4.11. Square wave function: (*a*) $L = 4$; (*b*) $L = 8$; (*c*) $L \to \infty$.

which has period $L > 2$. Fig. 4.11(a) shows the function when $L = 4$. If L is increased to 8, the function looks like the one shown in Fig. 4.11(b). As $L \to \infty$ we obtain a non-periodic function $f(x)$, as shown in Fig. 4.11(c):

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Example 4.6

Consider the periodic function $g_L(x)$ (Fig. 4.12(a)):

$$g_L(x) = e^{-|x|} \quad \text{when} \quad -L/2 < x < L/2.$$

As $L \to \infty$ we obtain a non-periodic function $g(x)$: $g(x) = \lim_{L\to\infty} g_L(x)$ (Fig. 4.12(b)).

By investigating the limit that is approached by a Fourier series as the period of the given function becomes infinite, a suitable representation for non-periodic functions can perhaps be obtained. To this end, let us write the Fourier series representing a periodic function $f(x)$ in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega x}, \tag{4.25}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega x}dx \tag{4.26}$$

where ω denotes $n\pi/L$

$$\omega = \frac{n\pi}{L}, \quad n \text{ positive or negative.} \tag{4.27}$$

The transition $L \to \infty$ is a little tricky since c_n apparently approaches zero, but these coefficients should not approach zero. We can ask for help from Eq. (4.27), from which we have

$$\Delta\omega = (\pi/L)\Delta n,$$

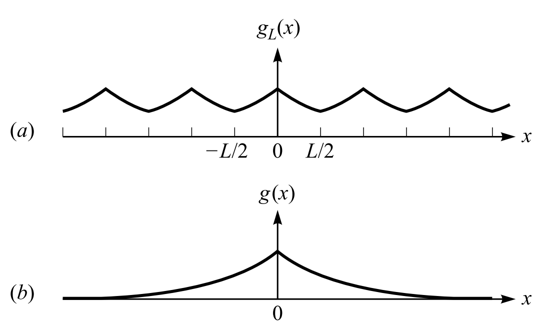

Figure 4.12. Sawtooth wave functions: (a) $-L/2 < x < L/2$; (b) $L \to \infty$.

and the 'adjacent' values of ω are obtained by setting $\Delta n = 1$, which corresponds to

$$(L/\pi)\Delta\omega = 1.$$

Then we can multiply each term of the Fourier series by $(L/\pi)\Delta\omega$ and obtain

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{L}{\pi} c_n\right) e^{i\omega x} \Delta\omega,$$

where

$$\frac{L}{\pi} c_n = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-i\omega x} dx.$$

The troublesome factor $1/L$ has disappeared. Switching completely to the ω notation and writing $(L/\pi)c_n = c_L(\omega)$, we obtain

$$c_L(\omega) = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-i\omega x} dx$$

and

$$f(x) = \sum_{L\omega/\pi=-\infty}^{\infty} c_L(\omega) e^{i\omega x} \Delta\omega.$$

In the limit as $L \to \infty$, the ωs are distributed continuously instead of discretely, $\Delta\omega \to d\omega$ and this sum is exactly the definition of an integral. Thus the last equations become

$$c(\omega) = \lim_{L\to\infty} c_L(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{4.28}$$

and

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} d\omega. \tag{4.29}$$

This set of formulas is known as the Fourier transformation, in somewhat different form. It is easy to put them in a symmetrical form by defining

$$g(\omega) = \sqrt{2\pi} c(-\omega),$$

then Eqs. (4.28) and (4.29) take the symmetrical form

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x') e^{-i\omega x'} dx', \tag{4.30}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} d\omega. \tag{4.31}$$

The function $g(\omega)$ is called the Fourier transform of $f(x)$ and is written $g(\omega) = F\{f(x)\}$. Eq. (4.31) is the inverse Fourier transform of $g(\omega)$ and is written $f(x) = F^{-1}\{g(\omega)\}$; sometimes it is also called the Fourier integral representation of $f(x)$. The exponential function $e^{-i\omega x}$ is sometimes called the kernel of transformation.

It is clear that $g(\omega)$ is defined only if $f(x)$ satisfies certain restrictions. For instance, $f(x)$ should be integrable in some finite region. In practice, this means that $f(x)$ has, at worst, jump discontinuities or mild infinite discontinuities. Also, the integral should converge at infinity. This would require that $f(x) \to 0$ as $x \to \pm\infty$.

A very common sufficient condition is the requirement that $f(x)$ is absolutely integrable. That is, the integral

$$\int_{-\infty}^{\infty} |f(x)| dx$$

exists. Since $|f(x)e^{-i\omega x}| = |f(x)|$, it follows that the integral for $g(\omega)$ is absolutely convergent; therefore it is convergent.

It is obvious that $g(\omega)$ is, in general, a complex function of the real variable ω. So if $f(x)$ is real, then

$$g(-\omega) = g^*(\omega).$$

There are two immediate corollaries to this property:

(1) $f(x)$ is even, $g(\omega)$ is real;
(2) if $f(x)$ is odd, $g(\omega)$ is purely imaginary.

Other, less symmetrical forms of the Fourier integral can be obtained by working directly with the sine and cosine series, instead of with the exponential functions.

Example 4.7
Consider the Gaussian probability function $f(x) = Ne^{-\alpha x^2}$, where N and α are constant. Find its Fourier transform $g(\omega)$, then graph $f(x)$ and $g(\omega)$.

Solution: Its Fourier transform is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-i\omega x} dx.$$

This integral can be simplified by a change of variable. First, we note that

$$-\alpha x^2 - i\omega x = -(x\sqrt{\alpha} + i\omega/2\sqrt{\alpha})^2 - \omega^2/4\alpha,$$

and then make the change of variable $x\sqrt{\alpha} + i\omega/2\sqrt{\alpha} = u$ to obtain

$$g(\omega) = \frac{N}{\sqrt{2\pi\alpha}} e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{N}{\sqrt{2\alpha}} e^{-\omega^2/4\alpha}.$$

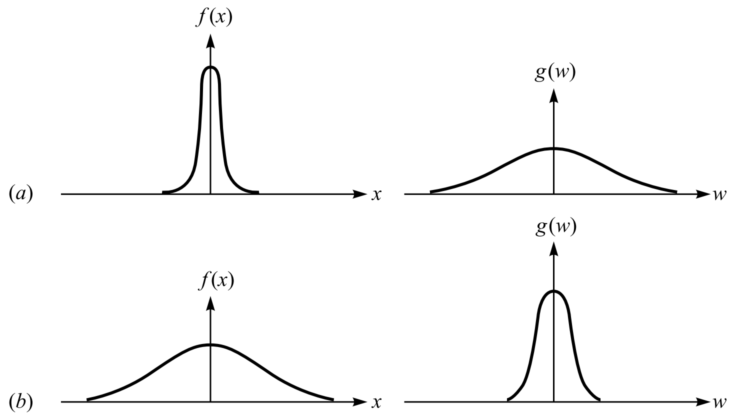

Figure 4.13. Gaussian probability function: (*a*) large α; (*b*) small α.

It is easy to see that $g(\omega)$ is also a Gaussian probability function with a peak at the origin, monotonically decreasing as $\omega \to \pm\infty$. Furthermore, for large α, $f(x)$ is sharply peaked but $g(\omega)$ is flattened, and vice versa as shown in Fig. 4.13. It is interesting to note that this is a general feature of Fourier transforms. We shall see later that in quantum mechanical applications it is related to the Heisenberg uncertainty principle.

The original function $f(x)$ can be retrieved from Eq. (4.31) which takes the form

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}d\omega &= \frac{1}{\sqrt{2\pi}}\frac{N}{\sqrt{2\alpha}}\int_{-\infty}^{\infty} e^{-\omega^2/4\alpha}e^{i\omega x}d\omega \\ &= \frac{1}{\sqrt{2\pi}}\frac{N}{\sqrt{2\alpha}}\int_{-\infty}^{\infty} e^{-\alpha'\omega^2}e^{-i\omega x'}d\omega\end{aligned}$$

in which we have set $\alpha' = 1/4\alpha$, and $x' = -x$. The last integral can be evaluated by the same technique, and we finally find

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}d\omega &= \frac{1}{\sqrt{2\pi}}\frac{N}{\sqrt{2\alpha}}\int_{-\infty}^{\infty} e^{-\alpha'\omega^2}e^{-i\omega x'}d\omega \\ &= \frac{N}{\sqrt{2\alpha}}\sqrt{2\alpha}e^{-\alpha x^2} \\ &= Ne^{-\alpha x^2} = f(x).\end{aligned}$$

Example 4.8

Given the box function which can represent a single pulse

$$f(x) = \begin{cases} 1 & |x|a \\ 0 & |x| > a \end{cases}$$

find the Fourier transform of $f(x)$, $g(\omega)$; then graph $f(x)$ and $g(\omega)$ for $a = 3$.

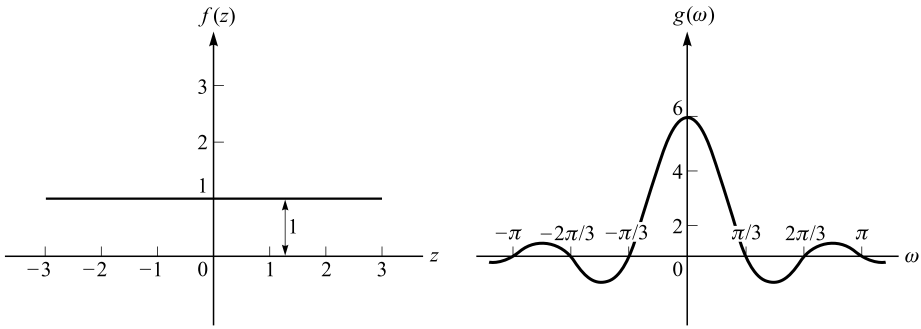

Figure 4.14. The box function.

Solution: The Fourier transform of $f(x)$ is, as shown in Fig. 4.14,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')e^{-i\omega x'} dx' = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)e^{-i\omega x'} dx' = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x'}}{-i\omega} \right|_{-a}^{a}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}, \quad \omega \neq 0.$$

For $\omega = 0$, we obtain $g(\omega) = \sqrt{2/\pi}a$.

The Fourier integral representation of $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \omega a}{\omega} e^{i\omega x} d\omega.$$

Now

$$\int_{-\infty}^{\infty} \frac{\sin \omega a}{\omega} e^{i\omega x} d\omega = \int_{-\infty}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} d\omega + i \int_{-\infty}^{\infty} \frac{\sin \omega a \sin \omega x}{\omega} d\omega.$$

The integrand in the second integral is odd and so the integral is zero. Thus we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)e^{i\omega x} d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} d\omega = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} d\omega;$$

the last step follows since the integrand is an even function of ω.

It is very difficult to evaluate the last integral. But a known property of $f(x)$ will help us. We know that $f(x)$ is equal to 1 for $|x| \leq a$, and equal to 0 for $|x| > a$. Thus we can write

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega a \cos \omega x}{\omega} d\omega = \begin{cases} 1 & |x|a \\ 0 & |x| > a \end{cases}$$

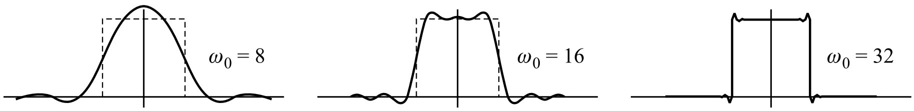

Figure 4.15. The Gibb's phenomenon.

Just as in Fourier series expansion, we also expect to observe Gibb's phenomenon in the case of Fourier integrals. Approximations to the Fourier integral are obtained by replacing ∞ by α:

$$\int_0^{\alpha} \frac{\sin\omega\cos\omega x}{\omega} d\omega,$$

where we have set $a = 1$. Fig. 4.15 shows oscillations near the points of discontinuity of $f(x)$. We might expect these oscillations to disappear as $\alpha \to \infty$, but they are just shifted closer to the points $x = \pm 1$.

Example 4.9

Consider now a harmonic wave of frequency ω_0, $e^{i\omega_0 t}$, which is chopped to a lifetime of $2T$ seconds (Fig. 4.16(a)):

$$f(t) = \begin{cases} e^{i\omega_0 t} & -T \le t \le T \\ 0 & |t| > 0 \end{cases}.$$

The chopping process will introduce many new frequencies in varying amounts, given by the Fourier transform. Then we have, according to Eq. (4.30),

$$g(\omega) = (2\pi)^{-1/2} \int_{-T}^{T} e^{i\omega_0 t} e^{-i\omega t} dt = (2\pi)^{-1/2} \int_{-T}^{T} e^{i(\omega_0 - \omega)t} dt$$
$$= (2\pi)^{-1/2} \frac{e^{i(\omega_0-\omega)t}}{i(\omega_0 - \omega)}\bigg|_{-T}^{T} = (2/\pi)^{1/2} T \frac{\sin(\omega_0 - \omega)T}{(\omega_0 - \omega)T}.$$

This function is plotted schematically in Fig. 4.16(b). (Note that $\lim_{x\to 0}(\sin x/x) = 1$.) The most striking aspect of this graph is that, although the principal contribution comes from the frequencies in the neighborhood of ω_0, an infinite number of frequencies are presented. Nature provides an example of this kind of chopping in the emission of photons during electronic and nuclear transitions in atoms. The light emitted from an atom consists of regular vibrations that last for a finite time of the order of 10^{-9} s or longer. When light is examined by a spectroscope (which measures the wavelengths and, hence, the frequencies) we find that there is an irreducible minimum frequency spread for each spectrum line. This is known as the natural line width of the radiation.

The relative percentage of frequencies, other than the basic one, present depends on the shape of the pulse, and the spread of frequencies depends on

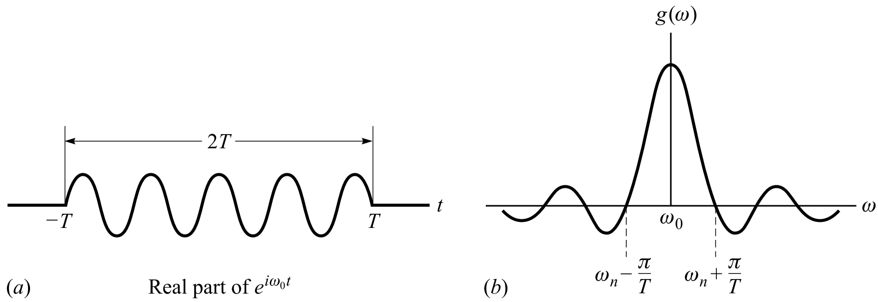

Figure 4.16. (*a*) A chopped harmonic wave $e^{i\omega_0 t}$ that lasts a finite time $2T$. (*b*) Fourier transform of $e^{(i\omega_0 t)}$, $|t| < T$, and 0 otherwise.

the time T of the duration of the pulse. As T becomes larger the central peak becomes higher and the width $\Delta\omega(= 2\pi/T)$ becomes smaller. Considering only the spread of frequencies in the central peak we have

$$\Delta\omega = 2\pi/T, \quad \text{or} \quad T\Delta = 1.$$

Multiplying by the Planck constant h and replacing T by Δt, we have the relation

$$\Delta t \Delta E = h. \tag{4.32}$$

A wave train that lasts a finite time also has a finite extension in space. Thus the radiation emitted by an atom in 10^{-9} s has an extension equal to $3 \times 10^8 \times 10^{-9} = 3 \times 10^{-1}$ m. A Fourier analysis of this pulse in the space domain will yield a graph identical to Fig. 4.11(*b*), with the wave numbers clustered around $k_0(= 2\pi/\lambda_0 = \omega_0/v)$. If the wave train is of length $2a$, the spread in wave number will be given by $a\Delta k = 2\pi$, as shown below. This time we are chopping an infinite plane wave front with a shutter such that the length of the packet is $2a$, where $2a = 2vT$, and $2T$ is the time interval that the shutter is open. Thus

$$\psi(x) = \begin{cases} e^{ik_0 x}, & -a \le x \le a \\ 0, & |x| > a \end{cases}.$$

Then

$$\begin{aligned} \phi(k) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx = (2\pi)^{-1/2} \int_{-a}^{a} \psi(x) e^{-ikx} dx \\ &= (2/\pi)^{1/2} a \frac{\sin(k_0 - k)a}{(k_0 - k)a}. \end{aligned}$$

This function is plotted in Fig. 4.17: it is identical to Fig. 4.16(*b*), but here it is the wave vector (or the momentum) that takes on a spread of values around k_0. The breadth of the central peak is $\Delta k = 2\pi/a$, or $a\Delta k = 2\pi$.

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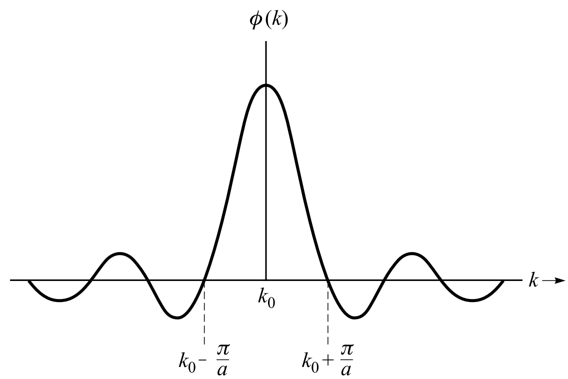

Figure 4.17. Fourier transform of e^{ikx}, $|x| \le a$.

Fourier sine and cosine transforms

If $f(x)$ is an odd function, the Fourier transforms reduce to

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x') \sin \omega x' dx', \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\omega) \sin \omega x d\omega. \tag{4.33a}$$

Similarly, if $f(x)$ is an even function, then we have Fourier cosine transformations:

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x') \cos \omega x' dx', \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\omega) \cos \omega x d\omega. \tag{4.33b}$$

To demonstrate these results, we first expand the exponential function on the right hand side of Eq. (4.30)

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x') e^{-i\omega x'} dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x') \cos \omega x' dx' - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x') \sin \omega x' dx'. \end{aligned}$$

If $f(x)$ is even, then $f(x) \cos \omega x$ is even and $f(x) \sin \omega x$ is odd. Thus the second integral on the right hand side of the last equation is zero and we have

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x') \cos \omega x' dx' = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x') \cos \omega x' dx';$$

$g(\omega)$ is an even function, since $g(-\omega) = g(\omega)$. Next from Eq. (4.31) we have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty g(\omega) e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty g(\omega) \cos \omega x d\omega + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty g(\omega) \sin \omega x d\omega. \end{aligned}$$

Since $g(\omega)$ is even, so $g(\omega)\sin\omega x$ is odd and the second integral on the right hand side of the last equation is zero, and we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\cos\omega x d\omega = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(\omega)\cos\omega x d\omega.$$

Similarly, we can prove Fourier sine transforms by replacing the cosine by the sine.

Heisenberg's uncertainty principle

We have demonstrated in above examples that if $f(x)$ is sharply peaked, then $g(\omega)$ is flattened, and vice versa. This is a general feature in the theory of Fourier transforms and has important consequences for all instances of wave propagation. In electronics we understand now why we use a wide-band amplification in order to reproduce a sharp pulse without distortion.

In quantum mechanical applications this general feature of the theory of Fourier transforms is related to the Heisenberg uncertainty principle. We saw in Example 4.9 that the spread of the Fourier transform in k space (Δk) times its spread in coordinate space (a) is equal to 2π ($a\Delta k \cong 2\pi$). This result is of special importance because of the connection between values of k and momentum p: $p = \hbar k$ (where $\hbar$ is the Planck constant h divided by 2π). A particle localized in space must be represented by a superposition of waves with different momenta. As a result, the position and momentum of a particle cannot be measured *simultaneously* with infinite precision; the product of 'uncertainty in the position determination' and 'uncertainty in the momentum determination' is governed by the relation $\Delta x\Delta p \cong h(a\hbar\Delta k \cong 2\pi\hbar = h$, or $\Delta x\Delta p \cong h, \Delta x = a)$. This statement is called Heisenberg's uncertainty principle. If position is known better, knowledge of the momentum must be unavoidably reduced proportionally, and vice versa. A complete knowledge of one, say k (and so p), is possible only when there is complete ignorance of the other. We can see this in physical terms. A wave with a unique value of k is infinitely long. A particle represented by an infinitely long wave (a free particle) cannot have a definite position, since the particle can be anywhere along its length. Hence the position uncertainty is infinite in order that the uncertainty in k is zero.

Equation (4.32) represents Heisenberg's uncertainty principle in a different form. It states that we cannot know with infinite precision the exact energy of a quantum system at every moment in time. In order to measure the energy of a quantum system with good accuracy, one must carry out such a measurement for a sufficiently long time. In other words, if the dynamical state exists only for a time of order Δt, then the energy of the state cannot be defined to a precision better than $h/\Delta t$.

We should not look upon the uncertainty principle as being merely an unfortunate limitation on our ability to know nature with infinite precision. We can use it to our advantage. For example, when combining the time–energy uncertainty relation with Einstein's mass–energy relation ($E = mc^2$) we obtain the relation $\Delta m \Delta t \cong h/c^2$. This result is very useful in our quest to understand the universe, in particular, the origin of matter.

Wave packets and group velocity

Energy (that is, a signal or information) is transmitted by groups of waves, not a single wave. Phase velocity may be greater than the speed of light c, 'group velocity' is always less than c. The wave groups with which energy is transmitted from place to place are called wave packets. Let us first consider a simple case where we have two waves φ_1 and φ_2: each has the same amplitude but differs slightly in frequency and wavelength,

$$\varphi_1(x,t) = A\cos(\omega t - kx),$$
$$\varphi_2(x,t) = A\cos[(\omega + \Delta\omega)t - (k + \Delta k)x],$$

where $\Delta\omega \ll \omega$ and $\Delta k \ll k$. Each represents a pure sinusoidal wave extending to infinite along the x-axis. Together they give a resultant wave

$$\begin{aligned}\varphi &= \varphi_1 + \varphi_2 \\ &= A\{\cos(\omega t - kx) + \cos[(\omega + \Delta\omega)t - (k + \Delta k)x]\}.\end{aligned}$$

Using the trigonometrical identity

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$

we can rewrite φ as

$$\begin{aligned}\varphi &= 2\cos\frac{2\omega t - 2kx + \Delta\omega t - \Delta kx}{2}\cos\frac{-\Delta\omega t + \Delta kx}{2} \\ &= 2\cos\tfrac{1}{2}(\Delta\omega t - \Delta kx)\cos(\omega t - kx).\end{aligned}$$

This represents an oscillation of the original frequency ω, but with a modulated amplitude as shown in Fig. 4.18. A given segment of the wave system, such as AB, can be regarded as a 'wave packet' and moves with a velocity v_g (not yet determined). This segment contains a large number of oscillations of the primary wave that moves with the velocity v. And the velocity v_g with which the modulated amplitude propagates is called the group velocity and can be determined by the requirement that the phase of the modulated amplitude be constant. Thus

$$v_g = dx/dt = \Delta\omega/\Delta k \to d\omega/dk.$$

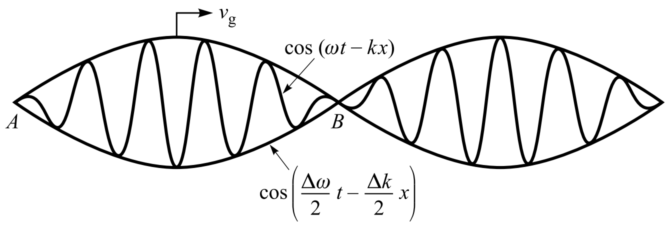

Figure 4.18. Superposition of two waves.

The modulation of the wave is repeated indefinitely in the case of superposition of two almost equal waves. We now use the Fourier technique to demonstrate that any isolated packet of oscillatory disturbance of frequency ω can be described in terms of a combination of infinite trains of frequencies distributed around ω. Let us first superpose a system of n waves

$$\psi(x, t) = \sum_{j=1}^{n} A_j e^{i(k_j x - \omega_j t)},$$

where A_j denotes the amplitudes of the individual waves. As n approaches infinity, the frequencies become continuously distributed. Thus we can replace the summation with an integration, and obtain

$$\psi(x, t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega t)} dk; \tag{4.34}$$

the amplitude $A(k)$ is often called the distribution function of the wave. For $\psi(x, t)$ to represent a wave packet traveling with a characteristic group velocity, it is necessary that the range of propagation vectors included in the superposition be fairly small. Thus, we assume that the amplitude $A(k) \neq 0$ only for a small range of values about a particular k_0 of k:

$$A(k) \neq 0, \quad k_0 - \varepsilon < k < k_0 + \varepsilon, \quad \varepsilon \ll k_0.$$

The behavior in time of the wave packet is determined by the way in which the angular frequency ω depends upon the wave number k: $\omega = \omega(k)$, known as the law of dispersion. If ω varies slowly with k, then $\omega(k)$ can be expanded in a power series about k_0:

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_0 (k - k_0) + \cdots = \omega_0 + \omega'(k - k_0) + O\left[(k - k_0)^2\right],$$

where

$$\omega_0 = \omega(k_0), \quad \text{and} \quad \omega' = \left.\frac{d\omega}{dk}\right|_0$$

and the subscript zero means 'evaluated' at $k = k_0$. Now the argument of the exponential in Eq. (4.34) can be rewritten as

$$\omega t - kx = (\omega_0 t - k_0 x) + \omega'(k - k_0)t - (k - k_0)x$$
$$= (\omega_0 t - k_0 x) + (k - k_0)(\omega' t - x)$$

and Eq. (4.34) becomes

$$\psi(x, t) = \exp[i(k_0 x - \omega_0 t)] \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k) \exp[i(k - k_0)(x - \omega' t)]dk. \quad (4.35)$$

If we take $k - k_0$ as the new integration variable y and assume $A(k)$ to be a slowly varying function of k in the integration interval 2ε, then Eq. (4.35) becomes

$$\psi(x, t) \cong \exp[i(k_0 x - \omega_0 t)] \int_{k_0-\varepsilon}^{k_0+\varepsilon} A(k_0 + y) \exp[i(x - \omega' t)y]dy.$$

Integration, transformation, and the approximation $A(k_0 + y) \cong A(k_0)$ lead to the result

$$\psi(x, t) = B(x, t) \exp[i(k_0 x - \omega_0 t)] \quad (4.36)$$

with

$$B(x, t) = 2A(k_0) \frac{\sin[\Delta k(x - \omega' t)]}{x - \omega' t}. \quad (4.37)$$

As the argument of the sine contains the small quantity $\Delta k, B(x, t)$ varies slowly depending on time t and coordinate x. Therefore, we can regard $B(x, t)$ as the small amplitude of an approximately monochromatic wave and $k_0 x - \omega_0 t$ as its phase. If we multiply the numerator and denominator on the right hand side of Eq. (4.37) by Δk and let

$$z = \Delta k(x - \omega' t)$$

then $B(x, t)$ becomes

$$B(x, t) = 2A(k_0)\Delta k \frac{\sin z}{z}$$

and we see that the variation in amplitude is determined by the factor sin $(z)/z$. This has the properties

$$\lim_{z\to 0} \frac{\sin z}{z} = 1 \quad \text{for} \quad z = 0$$

and

$$\frac{\sin z}{z} = 0 \quad \text{for} \quad z = \pm\pi, \pm 2\pi, \ldots.$$

If we further increase the absolute value of z, the function $\sin(z)/z$ runs alternately through maxima and minima, the function values of which are small compared with the principal maximum at $z = 0$, and quickly converges to zero. Therefore, we can conclude that superposition generates a wave packet whose amplitude is non-zero only in a finite region, and is described by $\sin(z)/z$ (see Fig. 4.19).

The modulating factor $\sin(z)/z$ of the amplitude assumes the maximum value 1 as $z \to 0$. Recall that $z = \Delta k(x - \omega' t)$, thus for $z = 0$, we have

$$x - \omega' t = 0,$$

which means that the maximum of the amplitude is a plane propagating with velocity

$$\frac{dx}{dt} = \omega' = \left.\frac{d\omega}{dk}\right|_0,$$

that is, ω' is the group velocity, the velocity of the whole wave packet.

The concept of a wave packet also plays an important role in quantum mechanics. The idea of associating a wave-like property with the electron and other material particles was first proposed by Louis Victor de Broglie (1892–1987) in 1925. His work was motivated by the mystery of the Bohr orbits. After Rutherford's successful α-particle scattering experiments, a planetary-type nuclear atom, with electrons orbiting around the nucleus, was in favor with most physicists. But, according to classical electromagnetic theory, a charge undergoing continuous centripetal acceleration emits electromagnetic radiation

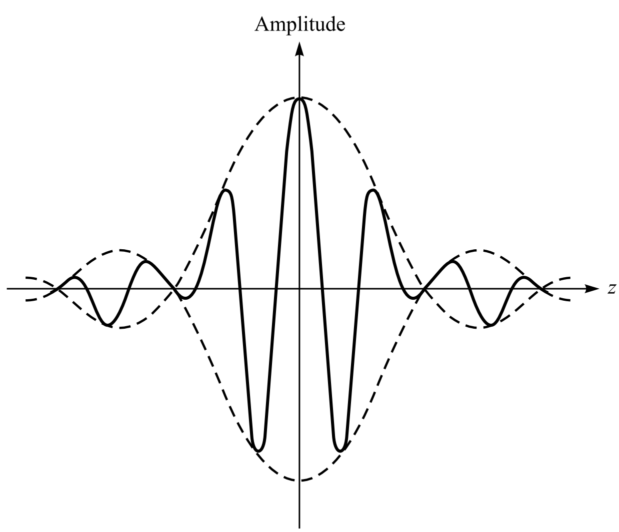

Figure 4.19. A wave packet.

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continuously and so the electron would lose energy continuously and it would spiral into the nucleus after just a fraction of a second. This does not occur. Furthermore, atoms do not radiate unless excited, and when radiation does occur its spectrum consists of discrete frequencies rather than the continuum of frequencies predicted by the classical electromagnetic theory. In 1913 Niels Bohr (1885–1962) proposed a theory which successfully explained the radiation spectra of the hydrogen atom. According to Bohr's postulates, an atom can exist in certain allowed stationary states without radiation. Only when an electron makes a transition between two allowed stationary states, does it emit or absorb radiation. The possible stationary states are those in which the angular momentum of the electron about the nucleus is quantized, that is, $mvr = n\hbar$, where v is the speed of the electron in the nth orbit and r is its radius. Bohr didn't clearly describe this quantum condition. De Broglie attempted to explain it by fitting a standing wave around the circumference of each orbit. Thus de Broglie proposed that $n\lambda = 2\pi r$, where λ is the wavelength associated with the nth orbit. Combining this with Bohr's quantum condition we immediately obtain

$$\lambda = \frac{h}{mv} = \frac{h}{p}.$$

De Broglie proposed that any material particle of total energy E and momentum p is accompanied by a wave whose wavelength is given by $\lambda = h/p$ and whose frequency is given by the Planck formula $\nu = E/h$. Today we call these waves de Broglie waves or matter waves. The physical nature of these matter waves was not clearly described by de Broglie, we shall not ask what these matter waves are – this is addressed in most textbooks on quantum mechanics. Let us ask just one question: what is the (phase) velocity of such a matter wave? If we denote this velocity by u, then

$$\begin{aligned} u = \lambda\nu = \frac{E}{p} &= \frac{1}{p}\sqrt{p^2c^2 + m_0^2c^4} \\ &= c\sqrt{1 + (m_0c/p)^2} = \frac{c^2}{v} \qquad \left(p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right), \end{aligned}$$

which shows that for a particle with $m_0 > 0$ the wave velocity u is always greater than c, the speed of light in a vacuum. Instead of individual waves, de Broglie suggested that we can think of particles inside a wave packet, synthesized from a number of individual waves of different frequencies, with the entire packet traveling with the particle velocity v.

De Broglie's matter wave idea is one of the cornerstones of quantum mechanics.

Heat conduction

We now consider an application of Fourier integrals in classical physics. A semi-infinite thin bar ($x \geq 0$), whose surface is insulated, has an initial temperature equal to $f(x)$. The temperature of the end $x = 0$ is suddenly dropped to and maintained at zero. The problem is to find the temperature $T(x, t)$ at any point x at time t. First we have to set up the boundary value problem for heat conduction, and then seek the general solution that will give the temperature $T(x, t)$ at any point x at time t.

Head conduction equation

To establish the equation for heat conduction in a conducting medium we need first to find the heat flux (the amount of heat per unit area per unit time) across a surface. Suppose we have a flat sheet of thickness Δn, which has temperature T on one side and $T + \Delta T$ on the other side (Fig. 4.20). The heat flux which flows from the side of high temperature to the side of low temperature is directly proportional to the difference in temperature ΔT and inversely proportional to the thickness Δn. That is, the heat flux from I to II is equal to

$$-K\frac{\Delta T}{\Delta n},$$

where K, the constant of proportionality, is called the thermal conductivity of the conducting medium. The minus sign is due to the fact that if $\Delta T > 0$ the heat actually flows from II to I. In the limit of $\Delta n \to 0$, the heat flux across from II to I can be written

$$-K\frac{\partial T}{\partial n} = -K\nabla T.$$

The quantity $\partial T/\partial n$ is called the gradient of T which in vector form is ∇T.

We are now ready to derive the equation for heat conduction. Let V be an arbitrary volume lying within the solid and bounded by surface S. The total

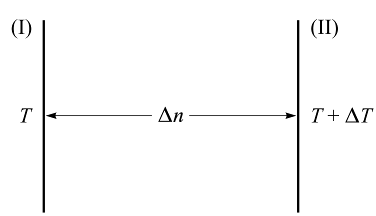

Figure 4.20. Heat flux through a thin sheet.

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amount of heat entering S per unit time is

$$\iint_S (K\nabla T)\cdot \hat{n}dS,$$

where $\hat{n}$ is an outward unit vector normal to element surface area dS. Using the divergence theorem, this can be written as

$$\iint_S (K\nabla T)\cdot \hat{n}dS = \iiint_V \nabla\cdot(K\nabla T)dV. \tag{4.38}$$

Now the heat contained in V is given by

$$\iiint_V c\rho T dV,$$

where c and ρ are respectively the specific heat capacity and density of the solid. Then the time rate of increase of heat is

$$\frac{\partial}{\partial t}\iiint_V c\rho T dV = \iiint_V c\rho \frac{\partial T}{\partial t} dV. \tag{4.39}$$

Equating the right hand sides of Eqs. (4.38) and (4.39) yields

$$\iiint_V \left[c\rho\frac{\partial T}{\partial t} - \nabla\cdot(K\nabla T)\right]dV = 0.$$

Since V is arbitrary, the integrand (assumed continuous) must be identically zero:

$$c\rho\frac{\partial T}{\partial t} = \nabla\cdot(K\nabla T)$$

or if K, c, ρ are constants

$$\frac{\partial T}{\partial t} = k\nabla\cdot\nabla T = k\nabla^2 T, \tag{4.40}$$

where $k = K/c\rho$. This is the required equation for heat conduction and was first developed by Fourier in 1822. For the semiinfinite thin bar, the boundary conditions are

$$T(x,0) = f(x),\ T(0,t) = 0, \qquad |T(x,t)| < M, \tag{4.41}$$

where the last condition means that the temperature must be bounded for physical reasons.

A solution of Eq. (4.40) can be obtained by separation of variables, that is by letting

$$T = X(x)H(t).$$

Then

$$XH' = kX''H \quad \text{or} \quad X''/X = H'/kH.$$

Each side must be a constant which we call $-\lambda^2$. (If we use $+\lambda^2$, the resulting solution does not satisfy the boundedness condition for real values of λ.) Then

$$X'' + \lambda^2 X = 0, \qquad H' + \lambda^2 k H = 0$$

with the solutions

$$X(x) = A_1 \cos \lambda x + B_1 \sin \lambda x, \qquad H(t) = C_1 e^{-k\lambda^2 t}.$$

A solution to Eq. (4.40) is thus given by

$$\begin{aligned} T(x,t) &= C_1 e^{-k\lambda^2 t}(A_1 \cos \lambda x + B_1 \sin \lambda x) \\ &= e^{-k\lambda^2 t}(A \cos \lambda x + B \sin \lambda x). \end{aligned}$$

From the second of the boundary conditions (4.41) we find $A = 0$ and so $T(x,t)$ reduces to

$$T(x,t) = Be^{-k\lambda^2 t} \sin \lambda x.$$

Since there is no restriction on the value of λ, we can replace B by a function $B(\lambda)$ and integrate over λ from 0 to ∞ and still have a solution:

$$T(x,t) = \int_0^\infty B(\lambda) e^{-k\lambda^2 t} \sin \lambda x d\lambda. \tag{4.42}$$

Using the first of boundary conditions (4.41) we find

$$f(x) = \int_0^\infty B(\lambda) \sin \lambda x d\lambda.$$

Then by the Fourier sine transform we find

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x) \sin \lambda x dx = \frac{2}{\pi}\int_0^\infty f(u) \sin \lambda u du$$

and the temperature distribution along the semiinfinite thin bar is

$$T(x,t) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u) e^{-k\lambda^2 t} \sin \lambda u \sin \lambda x d\lambda du. \tag{4.43}$$

Using the relation

$$\sin \lambda u \sin \lambda x = \tfrac{1}{2}[\cos \lambda(u-x) - \cos \lambda(u+x)],$$

Eq. (4.43) can be rewritten

$$\begin{aligned} T(x,t) &= \frac{1}{\pi}\int_0^\infty \int_0^\infty f(u) e^{-k\lambda^2 t}[\cos \lambda(u-x) - \cos \lambda(u+x)] d\lambda du \\ &= \frac{1}{\pi}\int_0^\infty f(u)\left[\int_0^\infty e^{-k\lambda^2 t} \cos \lambda(u-x) d\lambda - \int_0^\infty e^{-k\lambda^2 t} \cos \lambda(u+x) d\lambda\right] du. \end{aligned}$$

Using the integral

$$\int_0^\infty e^{-\alpha\lambda^2}\cos\beta\lambda d\lambda = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\beta^2/4\alpha},$$

we find

$$T(x,t) = \frac{1}{2\sqrt{\pi kt}}\left[\int_0^\infty f(u)e^{-(u-x)^2/4kt}du - \int_0^\infty f(u)e^{-(u+x)^2/4kt}du\right].$$

Letting $(u-x)/2\sqrt{kt} = w$ in the first integral and $(u+x)/2\sqrt{kt} = w$ in the second integral, we obtain

$$T(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_{-x/2\sqrt{kt}}^\infty e^{-w^2}f(2w\sqrt{kt}+x)dw - \int_{x/2\sqrt{kt}}^\infty e^{-w^2}f(2w\sqrt{kt}-x)dw\right].$$

Fourier transforms for functions of several variables

We can extend the development of Fourier transforms to a function of several variables, such as $f(x,y,z)$. If we first decompose the function into a Fourier integral with respect to x, we obtain

$$f(x,y,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \gamma(\omega_x,y,z)e^{i\omega_x x}d\omega_x,$$

where γ is the Fourier transform. Similarly, we can decompose the function with respect to y and z to obtain

$$f(x,y,z) = \frac{1}{(2\pi)^{2/3}}\int_{-\infty}^\infty g(\omega_x,\omega_y,\omega_z)e^{i(\omega_x x+\omega_y y+\omega_z z)}d\omega_x d\omega_y d\omega_z,$$

with

$$g(\omega_x,\omega_y,\omega_z) = \frac{1}{(2\pi)^{2/3}}\int_{-\infty}^\infty f(x,y,z)e^{-i(\omega_x x+\omega_y y+\omega_z z)}dxdydz.$$

We can regard $\omega_x,\omega_y,\omega_z$ as the components of a vector ω whose magnitude is

$$\omega = \sqrt{\omega_x^2+\omega_y^2+\omega_z^2},$$

then we express the above results in terms of the vector ω:

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{2/3}}\int_{-\infty}^\infty g(\boldsymbol{\omega})e^{i\boldsymbol{\omega}\cdot\mathbf{r}}d\boldsymbol{\omega}, \tag{4.44}$$

$$g(\boldsymbol{\omega}) = \frac{1}{(2\pi)^{2/3}}\int_{-\infty}^\infty f(\mathbf{r})e^{-(i\boldsymbol{\omega}\cdot\mathbf{r})}d\mathbf{r}. \tag{4.45}$$

The Fourier integral and the delta function

The delta function is a very useful tool in physics, but it is not a function in the usual mathematical sense. The need for this strange 'function' arises naturally from the Fourier integrals. Let us go back to Eqs. (4.30) and (4.31) and substitute $g(\omega)$ into $f(x)$; we then have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx' f(x') e^{i\omega(x-x')}.$$

Interchanging the order of integration gives

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega(x-x')}. \tag{4.46}$$

If the above equation holds for any function $f(x)$, then this tells us something remarkable about the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega(x-x')}$$

considered as a function of x'. It vanishes everywhere except at $x' = x$, and its integral with respect to x' over any interval including x is unity. That is, we may think of this function as having an infinitely high, infinitely narrow peak at $x = x'$. Such a strange function is called Dirac's delta function (first introduced by Paul A. M. Dirac):

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega(x-x')}. \tag{4.47}$$

Equation (4.46) then becomes

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x - x')dx'. \tag{4.48}$$

Equation (4.47) is an integral representation of the delta function. We summarize its properties below:

$$\delta(x - x') = 0, \quad \text{if } x' \neq x; \tag{4.49a}$$

$$\int_a^b \delta(x - x')dx' = \begin{cases} 0, & \text{if } x > b \text{ or } x < a \\ 1, & \text{if } a < x < b \end{cases}; \tag{4.49b}$$

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x - x')dx'. \tag{4.49c}$$

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It is often convenient to place the origin at the singular point, in which case the delta function may be written as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega x}. \tag{4.50}$$

To examine the behavior of the function for both small and large x, we use an alternative representation of this function obtained by integrating as follows:

$$\delta(x) = \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^{a} e^{i\omega x} d\omega = \lim_{a\to\infty}\frac{1}{2\pi}\left[\frac{e^{iax} - e^{-iax}}{ix}\right] = \lim_{a\to\infty}\frac{\sin ax}{\pi x}, \tag{4.51}$$

where a is positive and real. We see immediately that $\delta(-x) = \delta(x)$. To examine its behavior for small x, we consider the limit as x goes to zero:

$$\lim_{x\to 0}\frac{\sin ax}{\pi x} = \frac{a}{\pi}\lim_{x\to 0}\frac{\sin ax}{ax} = \frac{a}{\pi}.$$

Thus, $\delta(0) = \lim_{a\to\infty}(a/\pi) \to \infty$, or the amplitude becomes infinite at the singularity. For large $|x|$, we see that $\sin(ax)/x$ oscillates with period $2\pi/a$, and its amplitude falls off as $1/|x|$. But in the limit as a goes to infinity, the period becomes infinitesimally narrow so that the function approaches zero everywhere except for the infinite spike of infinitesimal width at the singularity. What is the integral of Eq. (4.51) over all space?

$$\int_{-\infty}^{\infty}\lim_{a\to\infty}\frac{\sin ax}{\pi x}dx = \lim_{a\to\infty}\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin ax}{\pi x}dx = \frac{2}{\pi}\frac{\pi}{2} = 1.$$

Thus, the delta function may be thought of as a spike function which has unit area but a non-zero amplitude at the point of singularity, where the amplitude becomes infinite. No ordinary mathematical function with these properties exists. How do we end up with such an improper function? It occurs because the change of order of integration in Eq. (4.46) is not permissible. In spite of this, the Dirac delta function is a most convenient function to use symbolically. For in applications the delta function always occurs under an integral sign. Carrying out this integration, using the formal properties of the delta function, is really equivalent to inverting the order of integration once more, thus getting back to a mathematically correct expression. Thus, using Eq. (4.49) we have

$$\int_{-\infty}^{\infty} f(x)\delta(x - x')dx = f(x'),$$

but, on substituting Eq. (4.47) for the delta function, the integral on the left hand side becomes

$$\int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{i\omega(x-x')}\right\}dx$$

or, using the property $\delta(-x) = \delta(x)$,

$$\int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(x-x')}\right\}dx$$

and *changing the order of integration*, we have

$$\int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega x}\right\}e^{i\omega x'}dx.$$

Comparing this expression with Eqs. (4.30) and (4.31), we see at once that this double integral is equal to $f(x')$, the correct mathematical expression.

It is important to keep in mind that the delta function cannot be the end result of a calculation and has meaning only so long as a subsequent integration over its argument is carried out.

We can easily verify the following most frequently required properties of the delta function:

If $a < b$

$$\int_a^b f(x)\delta(x - x')dx = \begin{cases} f(x'), & \text{if } a < x' < b \\ 0, & \text{if } x' < a \text{ or } x' < b \end{cases}, \tag{4.52a}$$

$$\delta(-x) = \delta(x), \tag{4.52b}$$

$$\delta'(x) = -\delta'(-x), \qquad \delta'(x) = d\delta(x)/dx, \tag{4.52c}$$

$$x\delta(x) = 0, \tag{4.52d}$$

$$\delta(ax) = a^{-1}\delta(x), \qquad a > 0, \tag{4.52e}$$

$$\delta(x^2 - a^2) = (2a)^{-1}[\delta(x-a) + \delta(x+a)], \quad a > 0, \tag{4.52f}$$

$$\int \delta(a-x)\delta(x-b)dx = \delta(a-b), \tag{4.52g}$$

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{4.52h}$$

Each of the first six of these listed properties can be established by multiplying both sides by a continuous, differentiable function $f(x)$ and then integrating over x. For example, multiplying $x\delta'(x)$ by $f(x)$ and integrating over x gives

$$\int f(x)x\delta'(x)dx = -\int \delta(x)\frac{d}{dx}[xf(x)]dx$$
$$= -\int \delta(x)\left[f(x) + xf'(x)\right]dx = -\int f(x)\delta(x)dx.$$

Thus $x\delta(x)$ has the same effect when it is a factor in an integrand as has $-\delta(x)$.

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Parseval's identity for Fourier integrals

We arrived earlier at Parseval's identity for Fourier series. An analogy exists for Fourier integrals. If $g(\alpha)$ and $G(\alpha)$ are Fourier transforms of $f(x)$ and $F(x)$ respectively, we can show that

$$\int_{-\infty}^{\infty} f(x)F^*(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha)G^*(\alpha)d\alpha, \tag{4.54}$$

where $F^*(x)$ is the complex conjugate of $F(x)$. In particular, if $F(x) = f(x)$ and hence $G(\alpha) = g(\alpha)$, then we have

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |g(\alpha)|d\alpha. \tag{4.54}$$

Equation (4.53), or the more general Eq. (4.54), is known as the Parseval's identity for Fourier integrals. Its proof is straightforward:

$$\int_{-\infty}^{\infty} f(x)F^*(x)dx = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\alpha)e^{-i\alpha x}d\alpha\right]$$

$$\times\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G^*(\alpha')e^{i\alpha' x}d\alpha'\right]dx$$

$$= \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\alpha' g(\alpha)G^*(\alpha')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(\alpha-\alpha')}dx\right]$$

$$= \int_{-\infty}^{\infty} d\alpha g(\alpha)\int_{-\infty}^{\infty} d\alpha' G^*(\alpha')\delta(\alpha'-\alpha) = \int_{-\infty}^{\infty} g(\alpha)G^*(\alpha)d\alpha.$$

Parseval's identity is very useful in understanding the physical interpretation of the transform function $g(\alpha)$ when the physical significance of $f(x)$ is known. The following example will show this.

Example 4.10

Consider the following function, as shown in Fig. 4.21, which might represent the current in an antenna, or the electric field in a radiated wave, or displacement of a damped harmonic oscillator:

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t/T}\sin\omega_0 t & t > 0 \end{cases}.$$

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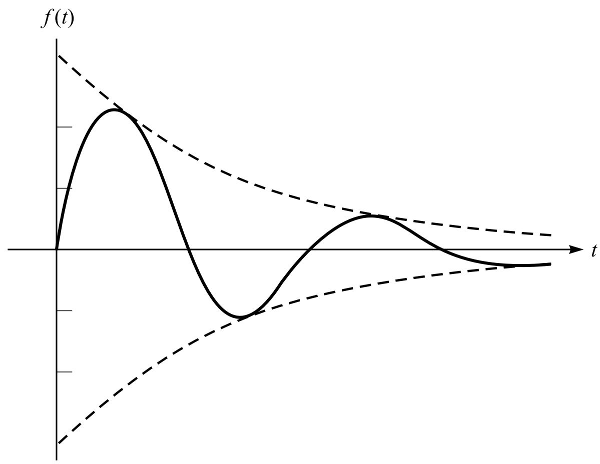

Figure 4.21. A damped sine wave.

Its Fourier transform $g(\omega)$ is

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t/T}e^{-i\omega t}\sin\omega_0 t dt$$
$$= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{\omega+\omega_0 - i/T} - \frac{1}{\omega - \omega_0 - i/T}\right).$$

If $f(t)$ is a radiated electric field, the radiated power is proportional to $|f(t)|^2$ and the total energy radiated is proportional to $\int_0^\infty |f(t)|^2 dt$. This is equal to $\int_0^\infty |g(\omega)|^2 d\omega$ by Parseval's identity. Then $|g(\omega)|^2$ must be the energy radiated per unit frequency interval.

Parseval's identity can be used to evaluate some definite integrals. As an example, let us revisit Example 4.8, where the given function is

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

and its Fourier transform is

$$g(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}.$$

By Parseval's identity, we have

$$\int_{-\infty}^{\infty}\{f(x)\}^2 dx = \int_{-\infty}^{\infty}\{g(\omega)\}^2 d\omega.$$

This is equivalent to

$$\int_{-a}^{a} (1)^2 dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{\sin^2 \omega a}{\omega^2} d\omega,$$

from which we find

$$\int_{0}^{\infty} \frac{2}{\pi} \frac{\sin^2 \omega a}{\omega^2} d\omega = \frac{\pi a}{2}.$$

The convolution theorem for Fourier transforms

The convolution of the functions $f(x)$ and $H(x)$, denoted by $f * H$, is defined by

$$f * H = \int_{-\infty}^{\infty} f(u)H(x-u)du. \tag{4.55}$$

If $g(\omega)$ and $G(\omega)$ are Fourier transforms of $f(x)$ and $H(x)$ respectively, we can show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)G(\omega)e^{i\omega x}d\omega = \int_{-\infty}^{\infty} f(u)H(x-u)du. \tag{4.56}$$

This is known as the convolution theorem for Fourier transforms. It means that the Fourier transform of the product $g(\omega)G(\omega)$, the left hand side of Eq. (55), is the convolution of the original function.

The proof is not difficult. We have, by definition of the Fourier transform,

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \qquad G(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x')e^{-i\omega x'}dx'.$$

Then

$$g(\omega)G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)H(x')e^{-i\omega(x+x')}dxdx'. \tag{4.57}$$

Let $x + x' = u$ in the double integral of Eq. (4.57) and we wish to transform from (x, x') to (x, u). We thus have

$$dxdx' = \frac{\partial(x, x')}{\partial(x, u)} dudx,$$

where the Jacobian of the transformation is

$$\frac{\partial(x, x')}{\partial(x, u)} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial u} \\[2ex] \dfrac{\partial x'}{\partial x} & \dfrac{\partial x'}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

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Thus Eq. (4.57) becomes

$$\begin{aligned} g(\omega)G(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)H(u-x)e^{-i\omega u}dxdu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}\left[\int_{-\infty}^{\infty} f(x)H(u-x)du\right]dx \\ &= F\left\{\int_{-\infty}^{\infty} f(x)H(u-x)du\right\} = F\{f * H\}. \end{aligned} \tag{4.58}$$

From this we have equivalently

$$f * H = F^{-1}\{g(\omega)G(\omega)\} = (1/2\pi)\int_{-\infty}^{\infty} e^{i\omega x} g(\omega)G(\omega),$$

which is Eq. (4.56).

Equation (4.58) can be rewritten as

$$F\{f\}F\{H\} = F\{f * H\} \qquad (g = F\{f\}, G = F\{H\}),$$

which states that *the Fourier transform of the convolution of $f(x)$ and $H(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $H(x)$*. This statement is often taken as the convolution theorem.

The convolution obeys the commutative, associative and distributive laws of algebra that is, if we have functions f_1, f_2, f_3 then

$$\left.\begin{aligned} &f_1 * f_2 = f_2 * f_1 && \text{commutative;} \\ &f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3 && \text{associative;} \\ &f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3 && \text{distributive.} \end{aligned}\right\} \tag{4.59}$$

It is not difficult to prove these relations. For example, to prove the commutative law, we first have

$$f_1 * f_2 \equiv \int_{-\infty}^{\infty} f_1(u)f_2(x-u)du.$$

Now let $x - u = v$, then

$$\begin{aligned} f_1 * f_2 &\equiv \int_{-\infty}^{\infty} f_1(u)f_2(x-u)du \\ &= \int_{-\infty}^{\infty} f_1(x-v)f_2(v)dv = f_2 * f_1. \end{aligned}$$

Example 4.11

Solve the integral equation $y(x) = f(x) + \int_{-\infty}^{\infty} y(u)r(x-u)du$, where $f(x)$ and $r(x)$ are given, and the Fourier transforms of $y(x)$, $f(x)$ and $r(x)$ exist.

Solution: Let us denote the Fourier transforms of $y(x)$, $f(x)$ and $r(x)$ by $Y(\omega), F(\omega)$, and $R(\omega)$ respectively. Taking the Fourier transform of both sides of the given integral equation, we have by the convolution theorem

$$Y(\omega) = F(\omega) + Y(\omega)R(\omega) \quad \text{or} \quad Y(\omega) = \frac{F(\omega)}{1 - R(\omega)}.$$

Calculations of Fourier transforms

Fourier transforms can often be used to transform a differential equation which is difficult to solve into a simpler equation that can be solved relatively easy. In order to use the transform methods to solve first- and second-order differential equations, the transforms of first- and second-order derivatives are needed. By taking the Fourier transform with respect to the variable x, we can show that

$$\left.\begin{aligned} (a)\ \ & F\left(\frac{\partial u}{\partial x}\right) = i\alpha F(u), \\ (b)\ \ & F\left(\frac{\partial^2 u}{\partial x^2}\right) = -\alpha^2 F(u), \\ (c)\ \ & F\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t} F(u). \end{aligned}\right\} \tag{4.60}$$

Proof: (a) By definition we have

$$F\left(\frac{\partial u}{\partial x}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\alpha x} dx,$$

where the factor $1/\sqrt{2\pi}$ has been dropped. Using integration by parts, we obtain

$$\begin{aligned} F\left(\frac{\partial u}{\partial x}\right) &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\alpha x} dx \\ &= u e^{-i\alpha x}\Big|_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} u e^{-i\alpha x} dx \\ &= i\alpha F(u). \end{aligned}$$

(b) Let $u = \partial v/\partial x$ in (a), then

$$F\left(\frac{\partial^2 v}{\partial x^2}\right) = i\alpha F\left(\frac{\partial v}{\partial x}\right) = (i\alpha)^2 F(v).$$

Now if we formally replace v by u we have

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = -\alpha^2 F(u),$$

provided that u and $\partial u/\partial x \to 0$ as $x \to \pm\infty$. In general, we can show that

$$F\left(\frac{\partial^n u}{\partial x^n}\right) = (i\alpha)^n F(u)$$

if $u, \partial u/\partial x, \ldots, \partial^{n-1}u/\partial x^{n-1} \to \pm\infty$ as $x \to \pm\infty$.

(c) By definition

$$F\left(\frac{\partial u}{\partial t}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\alpha x} dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-i\alpha x} dx = \frac{\partial}{\partial t} F(u).$$

Example 4.12

Solve the inhomogeneous differential equation

$$\left(\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right) f(x) = R(x), \; -\infty \le x \le \infty,$$

where p and q are constants.

Solution: We transform both sides

$$F\left\{\frac{d^2 f}{dx^2} + p\frac{df}{dx} + qf\right\} = [(i\alpha)^2 + p(i\alpha) + q]F\{f(x)\}$$
$$= F\{R(x)\}.$$

If we denote the Fourier transforms of $f(x)$ and $R(x)$ by $g(\alpha)$ and $G(\alpha)$, respectively,

$$F\{f(x)\} = g(\alpha), \quad F\{R(x)\} = G(\alpha),$$

we have

$$(-\alpha^2 + ip\alpha + q)g(\alpha) = G(\alpha), \quad \text{or} \quad g(\alpha) = G(\alpha)/(-\alpha^2 + ip\alpha + q)$$

and hence

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} g(\alpha) d\alpha$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} \frac{G(\alpha)}{-\alpha^2 + ip\alpha + q} d\alpha.$$

We will not gain anything if we do not know how to evaluate this complex integral. This is not a difficult problem in the theory of functions of complex variables (see Chapter 7).

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The delta function and the Green's function method

The Green's function method is a very useful technique in the solution of partial differential equations. It is usually used when boundary conditions, rather than initial conditions, are specified. To appreciate its usefulness, let us consider the inhomogeneous differential equation

$$L(x)f(x) - \lambda f(x) = R(x) \tag{4.61}$$

over a domain D, with L an arbitrary differential operator, and λ a given constant. Suppose we can expand $f(x)$ and $R(x)$ in eigenfunctions u_n of the operator $L(Lu_n = \lambda_n u_n)$:

$$f(x) = \sum_n c_n u_n(x), \qquad R(x) = \sum_n d_n u_n(x).$$

Substituting these into Eq. (4.61) we obtain

$$\sum_n c_n(\lambda_n - \lambda)u_n(x) = \sum_n d_n u_n(x).$$

Since the eigenfunctions $u_n(x)$ are linearly independent, we must have

$$c_n(\lambda_n - \lambda) = d_n \quad \text{or} \quad c_n = d_n/(\lambda_n - \lambda).$$

Moreover,

$$d_n = \int_D u_n^* R(x)dx.$$

Now we may write c_n as

$$c_n = \frac{1}{\lambda_n - \lambda}\int_D u_n^* R(x)dx,$$

therefore

$$f(x) = \sum_n \frac{u_n}{\lambda_n - \lambda}\int_D u_n^*(x')R(x')dx'.$$

This expression may be written in the form

$$f(x) = \int_D G(x, x')R(x')dx', \tag{4.62}$$

where $G(x, x')$ is given by

$$G(x, x') = \sum_n \frac{u_n(x)u_n^*(x')}{\lambda_n - \lambda} \tag{4.63}$$

and is called the Green's function. Some authors prefer to write $G(x, x'; \lambda)$ to emphasize the dependence of G on λ as well as on x and x'.

What is the differential equation obeyed by $G(x, x')$? Suppose $f(x')$ in Eq. (4.62) is taken to be $\delta(x' - x_0)$, then we obtain

$$f(x) = \int_D G(x, x')\delta(x' - x_0)dx = G(x, x_0).$$

Therefore $G(x, x')$ is the solution of

$$LG(x, x') - \lambda G(x, x') = \delta(x - x'), \tag{4.64}$$

subject to the appropriate boundary conditions. Eq. (4.64) shows clearly that *the Green's function is the solution of the problem for a unit point 'source'* $R(x) = \delta(x - x')$.

Example 4.13
Find the solution to the differential equation

$$\frac{d^2u}{dx^2} - k^2u = f(x) \tag{4.65}$$

on the interval $0 \le x \le l$, with $u(0) = u(l) = 0$, for a general function $f(x)$.

Solution: We first solve the differential equation which $G(x, x')$ obeys:

$$\frac{d^2G(x, x')}{dx^2} - k^2G(x, x') = \delta(x - x'). \tag{4.66}$$

For x equal to anything but x' (that is, for $x < x'$ or $x > x'$), $\delta(x - x') = 0$ and we have

$$\frac{d^2G_<(x, x')}{dx^2} - k^2G_<(x, x') = 0 \qquad (x < x'),$$

$$\frac{d^2G_>(x, x')}{dx^2} - k^2G_>(x, x') = 0 \qquad (x > x').$$

Therefore, for $x < x'$

$$G_< = Ae^{kx} + Be^{-kx}.$$

By the boundary condition $u(0) = 0$ we find $A + B = 0$, and $G_<$ reduces to

$$G_< = A(e^{kx} - e^{-kx}); \tag{4.67a}$$

similarly, for $x > x'$

$$G_> = Ce^{kx} + De^{-kx}.$$

By the boundary condition $u(l) = 0$ we find $Ce^{kl} + De^{-kl} = 0$, and $G_>$ can be rewritten as

$$G_> = C'[e^{k(x-l)} - e^{-k(x-l)}], \tag{4.67b}$$

where $C' = Ce^{kl}$.

How do we determine the constants A and C'? First, continuity of G at $x = x'$ gives

$$A(e^{kx} - e^{-kx}) = C'(e^{k(x-l)} - e^{-k(x-l)}). \tag{4.68}$$

A second constraint is obtained by integrating Eq. (4.61) from $x' - \varepsilon$ to $x' + \varepsilon$, where ε is infinitesimal:

$$\int_{x'-\varepsilon}^{x'+\varepsilon} \left[\frac{d^2G}{dx^2} - k^2 G\right] dx = \int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x - x')dx = 1. \tag{4.69}$$

But

$$\int_{x'-\varepsilon}^{x'+\varepsilon} k^2 G dx = k^2(G_> - G_<) = 0,$$

where the last step is required by the continuity of G. Accordingly, Eq. (4.64) reduces to

$$\int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d^2G}{dx^2} dx = \frac{dG_>}{dx} - \frac{dG_<}{dx} = 1. \tag{4.70}$$

Now

$$\left.\frac{dG_<}{dx}\right|_{x=x'} = Ak(e^{kx'} + e^{-kx'})$$

and

$$\left.\frac{dG_>}{dx}\right|_{x=x'} = C'k[e^{k(x'-l)} + e^{-k(x'-l)}].$$

Substituting these into Eq. (4.70) yields

$$C'k(e^{k(x'-l)} + e^{-k(x'-l)}) - Ak(e^{kx'} + e^{-kx'}) = 1. \tag{4.71}$$

We can solve Eqs. (4.68) and (4.71) for the constants A and C'. After some algebraic manipulation, the solution is

$$A = \frac{1}{2k}\frac{\sinh k(x'-l)}{\sinh kl}, \qquad C' = \frac{1}{2k}\frac{\sinh kx'}{\sinh kl}$$

and the Green's function is

$$G(x, x') = \frac{1}{k} \frac{\sinh kx \sinh k(x' - l)}{\sinh kl}, \tag{4.72}$$

which can be combined with $f(x)$ to obtain $u(x)$:

$$u(x) = \int_0^l G(x, x') f(x') dx'.$$

Problems

4.1 (*a*) Find the period of the function $f(x) = \cos(x/3) + \cos(x/4)$.
(*b*) Show that, if the function $f(t) = \cos\omega_1 t + \cos\omega_2 t$ is periodic with a period T, then the ratio ω_1/ω_2 must be a rational number.

4.2 Show that if $f(x + P) = f(x)$, then

$$\int_{a-P/2}^{a+P/2} f(x)dx = \int_{-P/2}^{P/2} f(x)dx, \qquad \int_P^{P+x} f(x)dx = \int_0^x f(x)dx.$$

4.3 (*a*) Using the result of Example 4.2, prove that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

(*b*) Using the result of Example 4.3, prove that

$$\frac{1}{1 \times 3} - \frac{1}{3 \times 5} + \frac{1}{5 \times 7} - + \cdots = \frac{\pi - 2}{4}.$$

4.4 Find the Fourier series which represents the function $f(x) = |x|$ in the interval $-\pi \le x \le \pi$.

4.5 Find the Fourier series which represents the function $f(x) = x$ in the interval $-\pi \le x \le \pi$.

4.6 Find the Fourier series which represents the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$.

4.7 Represent $f(x) = x, 0 < x < 2$, as: (*a*) in a half-range sine series, (*b*) a half-range cosine series.

4.8 Represent $f(x) = \sin x$, $0 < x < \pi$, as a Fourier cosine series.

4.9 (*a*) Show that the function $f(x)$ of period 2 which is equal to x on $(-1, 1)$ can be represented by the following Fourier series

$$-\frac{i}{\pi}\left(e^{i\pi x} - e^{-i\pi x} - \frac{1}{2}e^{2i\pi x} + \frac{1}{2}e^{-2i\pi x} + \frac{1}{3}e^{3i\pi x} - \frac{1}{3}e^{-3i\pi x} + \cdots\right).$$

(*b*) Write Parseval's identity corresponding to the Fourier series of (*a*).
(*c*) Determine from (*b*) the sum S of the series $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} 1/n^2$.

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4.10 Find the exponential form of the Fourier series of the function whose definition in one period is $f(x) = e^{-x}, -1 < x < 1$.

4.11 (*a*) Show that the set of functions

$$1, \quad \sin\frac{\pi x}{L}, \quad \cos\frac{\pi x}{L}, \quad \sin\frac{2\pi x}{L}, \quad \cos\frac{2\pi x}{L}, \quad \sin\frac{3\pi x}{L}, \quad \cos\frac{3\pi x}{L}, \ldots$$

form an orthogonal set in the interval $(-L, L)$.

(*b*) Determine the corresponding normalizing constants for the set in (*a*) so that the set is orthonormal in $(-L, L)$.

4.12 Express $f(x, y) = xy$ as a Fourier series for $0 \le x \le 1, 0 \le y \le 2$.

4.13 Steady-state heat conduction in a rectangular plate: Consider steady-state heat conduction in a flat plate having temperature values prescribed on the sides (Fig. 4.22). The boundary value problem modeling this is:

$$\frac{\partial^2 u}{\partial^2 x^2} + \frac{\partial^2 u}{\partial^2 y^2} = 0, \qquad 0 < x < \alpha, \quad 0 < y < \beta;$$

$$u(x, 0) = u(x, \beta) = 0, \qquad 0 < x < \alpha;$$

$$u(0, y) = 0, u(\alpha, y) = T, \qquad 0 < y < \beta.$$

Determine the temperature at any point of the plate.

4.14 Derive and solve the following eigenvalue problem which occurs in the theory of a vibrating square membrane whose sides, of length L, are kept fixed:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = 0,$$

$$w(0, y) = w(L, y) = 0 \quad (0 \le y \le L),$$

$$w(x, 0) = w(x, L) = 0 \quad (0 \le y \le L).$$

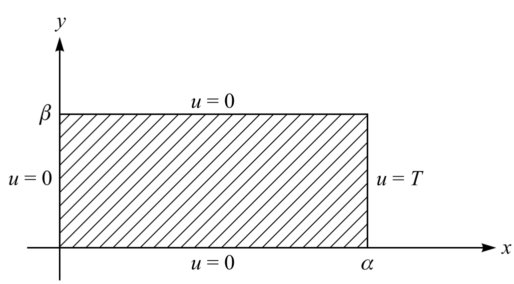

Figure 4.22. Flat plate with prescribed temperature.

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4.15 Show that the Fourier integral can be written in the form

$$f(x) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} f(x')\cos\omega(x - x')dx'.$$

4.16 Starting with the form obtained in Problem 4.15, show that the Fourier integral can be written in the form

$$f(x) = \int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx.$$

4.17 (*a*) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}.$$

(*b*) Evaluate

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx.$$

4.18 (*a*) Find the Fourier cosine transform of $f(x) = e^{-mx}, m > 0$.
(*b*) Use the result in (*a*) to show that

$$\int_0^\infty \frac{\cos px}{x^2 + \alpha^2}dx = \frac{\pi}{2\alpha}e^{-p\alpha} \quad (p > 0, \alpha > 0).$$

4.19 Solve the integral equation

$$\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1 - \alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}.$$

4.20 Find a bounded solution to Laplace's equation $\nabla^2 u(x, y) = 0$ for the half-plane $y > 0$ if u takes on the value of $f(x)$ on the x-axis:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(x, 0) = f(x), \qquad |u(x, y)| < M.$$

4.21 Show that the following two functions are valid representations of the delta function, where ε is positive and real:

(*a*) $\delta(x) = \dfrac{1}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\dfrac{1}{\sqrt{\varepsilon}}e^{-x^2/\varepsilon}$

(*b*) $\delta(x) = \dfrac{1}{\pi}\lim_{\varepsilon\to 0}\dfrac{\varepsilon}{x^2 + \varepsilon^2}$.

4.22 Verify the following properties of the delta function:

(*a*) $\delta(x) = \delta(-x)$,

(*b*) $x\delta(x) = 0$,

(*c*) $\delta'(-x) = -\delta'(x)$,

(*d*) $x\delta'(x) = -\delta(x)$,

(*e*) $c\delta(cx) = \delta(x), \quad c > 0$.

4.23 Solve the integral equation for $y(x)$

$$\int_{-\infty}^{\infty} \frac{y(u)du}{(x-u)^2 + a^2} = \frac{1}{x^2 + b^2} \qquad 0 < a < b.$$

4.24 Use Fourier transforms to solve the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = f(x), \qquad |u(x, t)| < M,$$

where $-\infty < x < \infty, t > 0$.

4.25 Obtain a solution to the equation of a driven harmonic oscillator

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t0 = R(t),$$

where β and ω_0 are positive and real constants.