6

Functions of a complex variable

The theory of functions of a complex variable is a basic part of mathematical analysis. It provides some of the very useful mathematical tools for physicists and engineers. In this chapter a brief introduction to complex variables is presented which is intended to acquaint the reader with at least the rudiments of this important subject.

Complex numbers

The number system as we know it today is a result of gradual development. The natural numbers (positive integers 1, 2, ...) were first used in counting. Negative integers and zero (that is, $0, -1, -2, \ldots$) then arose to permit solutions of equations such as $x + 3 = 2$. In order to solve equations such as $bx = a$ for all integers a and b where $b \neq 0$, rational numbers (or fractions) were introduced. Irrational numbers are numbers which cannot be expressed as a/b, with a and b integers and $b \neq 0$, such as $\sqrt{2} = 1.41423, \pi = 3.14159$

Rational and irrational numbers are all real numbers. However, the real number system is still incomplete. For example, there is no real number x which satisfies the algebraic equation $x^2 + 1 = 0$: $x = \sqrt{-1}$. The problem is that we do not know what to make of $\sqrt{-1}$ because there is no real number whose square is -1. Euler introduced the symbol $i = \sqrt{-1}$ in 1777 years later Gauss used the notation $a + ib$ to denote a complex number, where a and b are real numbers. Today, $i = \sqrt{-1}$ is called the unit imaginary number.

In terms of i, the answer to equation $x^2 + 1 = 0$ is $x = i$. It is postulated that i will behave like a real number in all manipulations involving addition and multiplication.

We now introduce a general complex number, in Cartesian form

$$z = x + iy \tag{6.1}$$

and refer to x and y as its real and imaginary parts and denote them by the symbols Re z and Im z, respectively. Thus if $z = -3 + 2i$, then Re $z = -3$ and Im $z = +2$.

A number with just $y \neq 0$ is called a pure imaginary number.

The complex conjugate, or briefly conjugate, of the complex number $z = x + iy$ is

$$z^* = x - iy \tag{6.2}$$

and is called 'z-star'. Sometimes we write it $\bar{z}$ and call it 'z-bar'. Complex conjugation can be viewed as the process of replacing i by $-i$ within the complex number.

Basic operations with complex numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs. Given two complex numbers z_1 and z_2 where $z_1 = a + ib, z_2 = c + id$, the basic rules obeyed by complex numbers are the following:

(1) Addition:

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d).$$

(2) Subtraction:

$$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d).$$

(3) Multiplication:

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad - bc).$$

(4) Division:

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Polar form of complex numbers

All real numbers can be visualized as points on a straight line (the x-axis). A complex number, containing two real numbers, can be represented by a point in a two-dimensional xy plane, known as the z plane or the complex plane (also known as the Gauss plane or Argand diagram). The complex variable $z = x + iy$ and its complex conjugation z^* are labeled in Fig. 6.1.

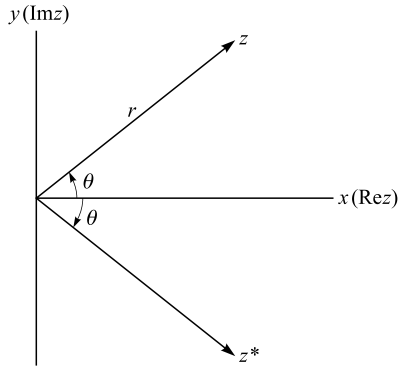

Figure 6.1. The complex plane.

The complex variable can also be represented by the plane polar coordinates (r, θ):

$$z = r(\cos\theta + i\sin\theta).$$

With the help of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can rewrite the last equation in polar form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad r = \sqrt{x^2 + y^2} = \sqrt{zz^*}. \tag{6.3}$$

r is called the modulus or absolute value of z, denoted by $|z|$ or mod z; and θ is called the phase or argument of z and it is denoted by arg z. For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leq \theta \leq 2\pi$. The absolute value of z has the following properties. If $z_1, z_2, \ldots, z_m$ are complex numbers, then we have:

(1) $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$.

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \ z_2 \neq 0$.

(3) $|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$.

(4) $|z_1 \pm z_2| \geq |z_1| - |z_2|$.

Complex numbers $z = re^{i\theta}$ with $r = 1$ have $|z| = 1$ and are called unimodular.

We may imagine them as lying on a circle of unit radius in the complex plane. Special points on this circle are

$$\theta = 0 \quad (1)$$
$$\theta = \pi/2 \quad (i)$$
$$\theta = \pi \quad (-1)$$
$$\theta = -\pi/2 \quad (-i).$$

The reader should know these points at all times.

Sometimes it is easier to use the polar form in manipulations. For example, to multiply two complex numbers, we multiply their moduli and add their phases; to divide, we divide by the modulus and subtract the phase of the denominator:

$$zz_1 = (re^{i\theta})(r_1 e^{i\theta_1}) = rr_1 e^{i(\theta+\theta_1)}, \qquad \frac{z}{z_1} = \frac{re^{i\theta}}{r_1 e^{i\theta_1}} = \frac{r}{r_1} e^{i(\theta-\theta_1)}.$$

On the other hand to add two complex numbers we have to go back to the Cartesian forms, add the components and revert to the polar form.

If we view a complex number z as a vector, then the multiplication of z by $e^{i\alpha}$ (where α is real) can be interpreted as a rotation of z counterclockwise through angle α; and we can consider $e^{i\alpha}$ as an operator which acts on z to produce this rotation. Similarly, the multiplication of two complex numbers represents a rotation and a change of length: $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$; the new complex number has length $r_1 r_2$ and phase $\theta_1 + \theta_2$.

Example 6.1
Find $(1+i)^8$.

Solution: We first write z in polar form: $z = 1 + i = r(\cos\theta + i\sin\theta)$, from which we find $r = \sqrt{2}, \theta = \pi/4$. Then

$$z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}.$$

Thus

$$(1+i)^8 = (\sqrt{2}e^{i\pi/4})^8 = 16e^{2\pi i} = 16.$$

Example 6.2
Show that

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{10} = \left(\frac{2e^{\pi i/3}}{2e^{-\pi i/3}}\right)^{10} = \left(e^{2\pi i/3}\right)^{10} = e^{20\pi i/3}$$

$$= e^{6\pi i}e^{2\pi i/3} = 1[\cos(2\pi/3) + i\sin(2\pi/3)] = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

De Moivre's theorem and roots of complex numbers

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

A generalization of this leads to

$$\begin{aligned} z_1 z_2 \cdots z_n &= r_1 r_2 \cdots r_n e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \\ &= r_1 r_2 \cdots r_n[\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)]; \end{aligned}$$

if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = (re^{i\theta})^n = r^n[\cos(n\theta) + i\sin(n\theta)],$$

from which it follows that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta), \qquad (6.4)$$

a result known as De Moivre's theorem. Thus we now have a general rule for calculating the nth power of a complex number z. We first write z in polar form $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n[\cos n\theta + i\sin n\theta]. \qquad (6.5)$$

The general rule for calculating the nth root of a complex number can now be derived without difficulty. A number w is called an nth root of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. If $z = r(\cos\theta + i\sin\theta)$, then the complex number

$$w_0 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

is definitely the nth root of z because $w_0^n = z$. But the numbers

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \qquad k = 1, 2, \ldots, (n-1),$$

are also nth roots of z because $w_k^n = z$. Thus the general rule for calculating the nth root of a complex number is

$$w = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \qquad k = 0, 1, 2, \ldots, (n-1). \qquad (6.6)$$

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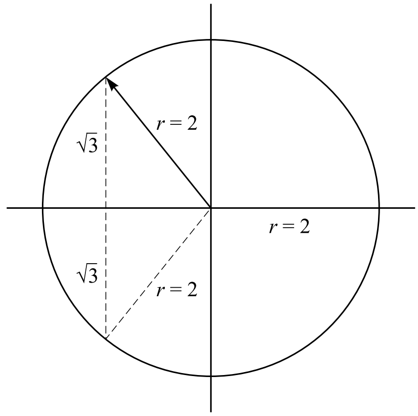

Figure 6.2. The cube roots of 8.

It is customary to call the number corresponding to $k = 0$ (that is, w_0) the principal root of z.

The nth roots of a complex number z are always located at the vertices of a regular polygon of n sides inscribed in a circle of radius $\sqrt[n]{r}$ about the origin.

Example 6.3
Find the cube roots of 8.

Solution: In this case $z = 8 + i0 = r(\cos\theta + i\sin\theta), r = 2$ and the principal argument $\theta = 0$. Formula (6.6) then yields

$$\sqrt[3]{8} = 2\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \qquad k = 0, 1, 2.$$

These roots are plotted in Fig. 6.2:

$$\begin{aligned} &2 && (k = 0, \theta = 0^\circ), \\ &-1 + i\sqrt{3} && (k = 1, \theta = 120^\circ), \\ &-1 - i\sqrt{3} && (k = 2, \theta = 240^\circ). \end{aligned}$$

Functions of a complex variable

Complex numbers $z = x + iy$ become variables if x or y (or both) vary. Then functions of a complex variable may be formed. If to each value which a complex variable z can assume there corresponds one or more values of a complex variable w, we say that w is a function of z and write $w = f(z)$ or $w = g(z)$, etc. The variable z is sometimes called an independent variable, and then w is a dependent

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variable. If only one value of w corresponds to each value of z, we say that w is a single-valued function of z or that $f(z)$ is single-valued; and if more than one value of w corresponds to each value of z, w is then a multiple-valued function of z. For example, $w = z^2$ is a single-valued function of z, but $w = \sqrt{z}$ is a double-valued function of z. In this chapter, whenever we speak of a function we shall mean a single-valued function, unless otherwise stated.

Mapping

Note that w is also a complex variable and so can be written in the form

$$w = u + iv = f(x + iy), \tag{6.7}$$

where u and v are real. By equating real and imaginary parts this is seen to be equivalent to

$$u = u(x, y), \quad v = v(x, y). \tag{6.8}$$

If $w = f(z)$ is a single-valued function of z, then to each point of the complex z plane, there corresponds a point in the complex w plane. If $f(z)$ is multiple-valued, a point in the z plane is mapped in general into more than one point. The following two examples show the idea of mapping clearly.

Example 6.4
Map $w = z^2 = r^2 e^{2i\theta}$.

Solution: This is single-valued function. The mapping is unique, but not one-to-one. It is a two-to-one mapping, since z and $-z$ give the same square. For example as shown in Fig. 6.3, $z = -2 + i$ and $z = 2 - i$ are mapped to the same point $w = 3 - 4i$; and $z = 1 - 3i$ and $-1 + 3i$ are mapped into the same point $w = -8 - 6i$.

The line joining the points $P(-2, 1)$ and $Q(1, -3)$ in the z-plane is mapped by $w = z^2$ into a curve joining the image points $P'(3, -4)$ and $Q'(-8, -6)$. It is not

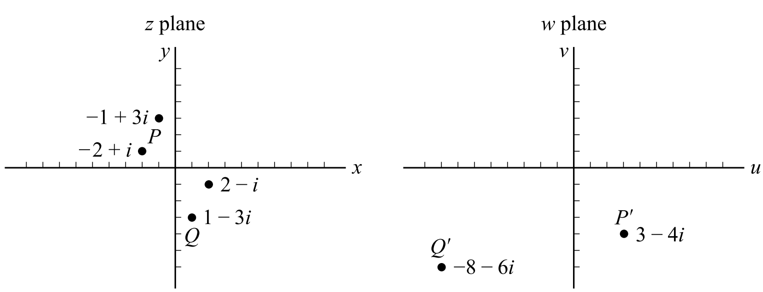

Figure 6.3. The mapping function $w = z^2$.

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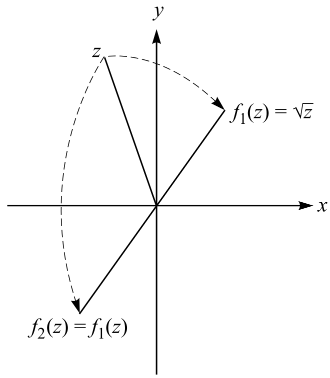

Figure 6.4. The mapping function $w = \sqrt{z}$.

very difficult to determine the equation of this curve. We first need the equation of the line joining P and Q in the z plane. The parametric equations of the line joining P and Q are given by

$$\frac{x-(-2)}{1-(-2)} = \frac{y-1}{-3-1} = t \quad \text{or} \quad x = 3t-2, y = 1-4t.$$

The equation of the line PQ is then given by $z = 3t - 2 + i(1-4t)$. The curve in the w plane into which the line PQ is mapped has the equation

$$w = z^2 = [3t - 2 + i(1-4t)]^2 = 3 - 4t - 7t^2 + i(-4 + 22t - 24t^2),$$

from which we obtain

$$u = 3 - 4t - 7t^2, \qquad v = -4 + 22t - 24t^2.$$

By assigning various values to the parameter t, this curve may be graphed.

Sometimes it is convenient to superimpose the z and w planes. Then the images of various points are located on the same plane and the function $w = f(z)$ may be said to transform the complex plane to itself (or a part of itself).

Example 6.5
Map $w = f(z) = \sqrt{z}, z = re^{i\theta}$.

Solution: There are two square roots: $f_1(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$, $f_2 = -f_1 = \sqrt{r}e^{i(\theta+2\pi)/2}$. The function is double-valued, and the mapping is one-to-two. This is shown in Fig. 6.4, where for simplicity we have used the same complex plane for both z and $w = f(z)$.

Branch lines and Riemann surfaces

We now take a close look at the function $w = \sqrt{z}$ of Example 6.5. Suppose we allow z to make a complete counterclockwise motion around the origin starting from point

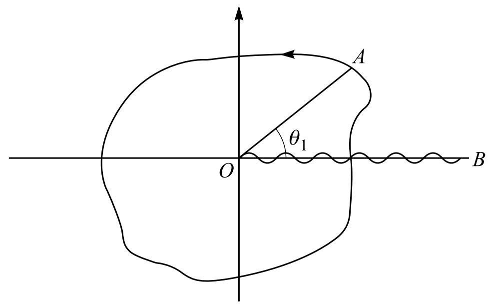

Figure 6.5. Branch cut for the function $w = \sqrt{z}$.

A, as shown in Fig. 6.5. At A, $\theta = \theta_1$ and $w = \sqrt{r}e^{i\theta/2}$. After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}e^{i(\theta+2\pi)/2} = -\sqrt{r}e^{i\theta/2}$. However, by making a second complete circuit back to A, $\theta = \theta_1 + 4\pi$, and so $w = \sqrt{r}e^{i(\theta+4\pi)/2} = \sqrt{r}e^{i\theta/2}$; that is, we obtain the same value of w with which we started.

We can describe the above by stating that if $0 \leq \theta < 2\pi$ we are on one branch of the multiple-valued function $\sqrt{z}$, while if $2\pi \leq \theta < 4\pi$ we are on the other branch of the function. It is clear that each branch of the function is single-valued. In order to keep the function single-valued, we set up an artificial barrier such as OB (the wavy line) which we agree not to cross. This artificial barrier is called a branch line or branch cut, point O is called a branch point. Any other line from O can be used for a branch line.

Riemann (George Friedrich Bernhard Riemann, 1826–1866) suggested another way to achieve the purpose of the branch line described above. Imagine the z plane consists of two sheets superimposed on each other. We now cut the two sheets along OB and join the lower edge of the bottom sheet to the upper edge of the top sheet. Then on starting in the bottom sheet and making one complete circuit about O we arrive in the top sheet. We must now imagine the other cut edges to be joined together (independent of the first join and actually disregarding its existence) so that by continuing the circuit we go from the top sheet back to the bottom sheet.

The collection of two sheets is called a Riemann surface corresponding to the function $\sqrt{z}$. Each sheet corresponds to a branch of the function and on each sheet the function is singled-valued. The concept of Riemann surfaces has the advantage that the various values of multiple-valued functions are obtained in a continuous fashion.

The differential calculus of functions of a complex variable

Limits and continuity

The definitions of limits and continuity for functions of a complex variable are similar to those for a real variable. We say that $f(z)$ has limit w_0 as z approaches

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z_0, which is written as

$$\lim_{z \to z_0} f(z) = w_0, \tag{6.9}$$

if

(a) $f(z)$ is defined and single-valued in a neighborhood of $z = z_0$, with the possible exception of the point z_0 itself; and
(b) given any positive number ε (however small), there exists a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

The limit must be independent of the manner in which z approaches z_0.

Example 6.6
(a) If $f(z) = z^2$, prove that $\lim_{z \to z_0}, f(z) = z_0^2$
(b) Find $\lim_{z \to z_0} f(z)$ if

$$f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}.$$

Solution: (a) We must show that given any $\varepsilon > 0$ we can find δ (depending in general on ε) such that $|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Now if $\delta \leq 1$, then $0 < |z - z_0| < \delta$ implies that

$$|z - z_0||z + z_0| < \delta|z + z_0| = \delta|z - z_0 + 2z_0|,$$

$$\left|z^2 - z_0^2\right| < \delta(|z - z_0| + 2|z_0|) < \delta(1 + 2|z_0|).$$

Taking δ as 1 or $\varepsilon/(1 + 2|z_0|)$, whichever is smaller, we then have $|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$, and the required result is proved.

(b) There is no difference between this problem and that in part (a), since in both cases we exclude $z = z_0$ from consideration. Hence $\lim_{z \to z_0} f(z) = z_0^2$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing to do with the value of $f(z)$ at z_0.

A function $f(z)$ is said to be continuous at z_0 if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. This implies three conditions that must be met in order that $f(z)$ be continuous at $z = z_0$:

(1) $\lim_{z \to z_0} f(z) = w_0$ must exist;
(2) $f(z_0)$ must exist, that is, $f(z)$ is defined at z_0;
(3) $w_0 = f(z_0)$.

For example, complex polynomials, $\alpha_0 + \alpha_1 z^1 + \alpha_2 z^2 + \alpha_n z^n$ (where α_i may be complex), are continuous everywhere. Quotients of polynomials are continuous whenever the denominator does not vanish. The following example provides further illustration.

A function $f(z)$ is said to be continuous in a region R of the z plane if it is continuous at all points of R.

Points in the z plane where $f(z)$ fails to be continuous are called discontinuities of $f(z)$, and $f(z)$ is said to be discontinuous at these points. If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call the point z_0 a removable discontinuity, since by redefining $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

To examine the continuity of $f(z)$ at $z = \infty$, we let $z = 1/w$ and examine the continuity of $f(1/w)$ at $w = 0$.

Derivatives and analytic functions

Given a continuous, single-valued function of a complex variable $f(z)$ in some region R of the z plane, the derivative $f'(z)(\equiv df/dz)$ at some fixed point z_0 in R is defined as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \tag{6.10}$$

provided the limit exists independently of the manner in which $\Delta z \to 0$. Here $\Delta z = z - z_0$, and z is any point of some neighborhood of z_0. If $f'(z)$ exists at z_0 and every point z in some neighborhood of z_0, then $f(z)$ is said to be analytic at z_0. And $f(z)$ is analytic in a region R of the complex z plane if it is analytic at every point in R.

In order to be analytic, $f(z)$ must be single-valued and continuous. It is straightforward to see this. In view of Eq. (6.10), whenever $f'(z_0)$ exists, then

$$\lim_{\Delta z \to 0} [f(z_0 + \Delta z) - f(z_0)] = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \lim_{\Delta z \to 0} \Delta z = 0$$

that is,

$$\lim_{\Delta z \to 0} f(z) = f(z_0).$$

Thus f is necessarily continuous at any point z_0 where its derivative exists. But the converse is not necessarily true, as the following example shows.

Example 6.7

The function $f(z) = z^*$ is continuous at z_0, but dz^*/dz does not exist anywhere. By definition,

$$\frac{dz^*}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^* - z^*}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \frac{(x + iy + \Delta x + i\Delta y)^* - (x + iy)^*}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y} = \lim_{\Delta x, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \Delta x / \Delta x = 1$. On the other hand, if $\Delta x = 0$, the required limit is -1. Then since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist and so $f(z) = z^*$ is non-analytic everywhere.

Example 6.8
Given $f(z) = 2z^2 - 1$, find $f'(z)$ at $z_0 = 1 - i$.

Solution:

$$\begin{aligned} f'(z_0) = f'(1-i) &= \lim_{z \to 1-i} \frac{(2z^2 - 1) - [2(1-i)^2 - 1]}{z - (1-i)} \\ &= \lim_{z \to 1-i} \frac{2[z - (1-i)][z + (1-i)]}{z - (1-i)} \\ &= \lim_{z \to 1-i} 2[z + (1-i)] = 4(1-i). \end{aligned}$$

The rules for differentiating sums, products, and quotients are, in general, the same for complex functions as for real-valued functions. That is, if $f'(z_0)$ and $g'(z_0)$ exist, then:

(1) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$;
(2) $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$;
(3) $\left(\dfrac{f}{g}\right)'(z_0) = \dfrac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \qquad \text{if } g'(z_0) \neq 0.$

The Cauchy–Riemann conditions

We call $f(z)$ analytic at z_0, if $f'(z)$ exists for all z in some δ neighborhood of z_0; and $f(z)$ is analytic in a region R if it is analytic at every point of R. Cauchy and Riemann provided us with a simple but extremely important test for the analyticity of $f(z)$. To deduce the Cauchy–Riemann conditions for the analyticity of $f(z)$, let us return to Eq. (6.10):

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

If we write $f(z) = u(x, y) + iv(x, y)$, this becomes

$$f'(z) = \lim_{\Delta x, \Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) - u(x, y) + i(\text{same for } v)}{\Delta x + i\Delta y}.$$

There are of course an infinite number of ways to approach a point z on a two-dimensional surface. Let us consider two possible approaches – along x and along

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y. Suppose we first take the x route, so y is fixed as we change x, that is, $\Delta y = 0$ and $\Delta x \to 0$, and we have

$$f'(z) = \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

We next take the y route, and we have

$$f'(z) = \lim_{\Delta y \to 0}\left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}\right] = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Now $f(z)$ cannot possibly be analytic unless the two derivatives are identical. Thus a necessary condition for $f(z)$ to be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y},$$

from which we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6.11}$$

These are the Cauchy–Riemann conditions, named after the French mathematician A. L. Cauchy (1789–1857) who discovered them, and the German mathematician Riemann who made them fundamental in his development of the theory of analytic functions. Thus if the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R, then $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann conditions at all points of R.

Example 6.9

If $f(z) = z^2 = x^2 - y^2 + 2ixy$, then $f'(z)$ exists for all $z: f'(z) = 2z$, and

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Thus, the Cauchy–Riemann equations (6.11) hold in this example at all points z.

We can also find examples in which $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann conditions (6.11) at $z = z_0$, but $f'(z_0)$ doesn't exist. One such example is the following:

$$f(z) = u(x,y) + iv(x,y) = \begin{cases} z^5/|z|^4 & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}.$$

The reader can show that $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann conditions (6.11) at $z = 0$, but that $f'(0)$ does not exist. Thus $f(z)$ is not analytic at $z = 0$. The proof is straightforward, but very tedious.

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However, the Cauchy–Riemann conditions do imply analyticity provided an additional hypothesis is added:

> Given $f(z) = u(x, y) + iv(x, y)$, if $u(x, y)$ and $v(x, y)$ are continuous with continuous first partial derivatives and satisfy the Cauchy–Riemann conditions (11) at all points in a region R, then $f(z)$ is analytic in R.

To prove this, we need the following result from the calculus of real-valued functions of two variables: If $h(x, y), \partial h/\partial x$, and $\partial h/\partial y$ are continuous in some region R about (x_0, y_0), then there exists a function $H(\Delta x, \Delta y)$ such that $H(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ and

$$h(x_0 + \Delta x, y_0 + \Delta y) - h(x_0, y_0) = \frac{\partial h(x_0, y_0)}{\partial x}\Delta x + \frac{\partial h(x_0, y_0)}{\partial \mathrm{y}}\Delta y + H(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Let us return to

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

where z_0 is any point in region R and $\Delta z = \Delta x + i\Delta y$. Now we can write

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] \\ &\quad + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)] \\ &= \frac{\partial u(x_0 y_0)}{\partial x}\Delta x + \frac{\partial u(x_0 y_0)}{\partial y}\Delta y + H(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &\quad + i\left[\frac{\partial v(x_0 y_0)}{\partial x}\Delta x + \frac{\partial v(x_0 y_0)}{\partial y}\Delta y \right. \\ &\quad \left. + G(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}\right], \end{aligned}$$

where $H(\Delta x, \Delta y) \to 0$ and $G(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Using the Cauchy–Riemann conditions and some algebraic manipulation we obtain

$$f(z_0 + \Delta z) - f(z_0) = \left[\frac{\partial u(x_0, y_0)}{\partial x} + i\frac{\partial v(x_0, y_0)}{\partial x}\right](\Delta x + i\Delta y) + [H(\Delta x, \Delta y) + iG(\Delta x, \Delta y)]\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

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and

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\partial u(x_0,y_0)}{\partial x}+i\frac{\partial v(x_0,y_0)}{\partial x}$$
$$+[H(\Delta x\cdot\Delta y)+iG(\Delta x\cdot\Delta y)]\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y}.$$

But

$$\left|\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x+i\Delta y}\right|=1.$$

Thus, as $\Delta z\to 0$, we have $(\Delta x,\Delta y)\to(0,0)$ and

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\partial u(x_0,y_0)}{\partial x}+i\frac{\partial v(x_0,y_0)}{\partial x},$$

which shows that the limit and so $f'(z_0)$ exist. Since $f(z)$ is differentiable at all points in region R, $f(z)$ is analytic at z_0 which is any point in R.

The Cauchy–Riemann equations turn out to be both necessary and sufficient conditions that $f(z)=u(x,y)+iv(x,y)$ be analytic. Analytic functions are also called regular or holomorphic functions. If $f(z)$ is analytic everywhere in the finite z complex plane, it is called an entire function. A function $f(z)$ is said to be singular at $z=z_0$, if it is not differentiable there; the point z_0 is called a singular point of $f(z)$.

Harmonic functions

If $f(z)=u(x,y)+iv(x,y)$ is analytic in some region of the z plane, then at every point of the region the Cauchy–Riemann conditions are satisfied:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad\text{and}\quad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

and therefore

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y},\quad\text{and}\quad\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x},$$

provided these second derivatives exist. In fact, one can show that if $f(z)$ is analytic in some region R, all its derivatives exist and are continuous in R. Equating the two cross terms, we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{6.12a}$$

throughout the region R.

Similarly, by differentiating the first of the Cauch–Riemann equations with respect to y, the second with respect to x, and subtracting we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{6.12b}$$

Eqs. (6.12a) and (6.12b) are Laplace's partial differential equations in two independent variables x and y. Any function that has continuous partial derivatives of second order and that satisfies Laplace's equation is called a harmonic function.

We have shown that if $f(z) = u(x, y) + iv(x, y)$ is analytic, then both u and v are harmonic functions. They are called conjugate harmonic functions. This is a different use of the word conjugate from that employed in determining z^*.

Given one of two conjugate harmonic functions, the Cauchy–Riemann equations (6.11) can be used to find the other.

Singular points

A point at which $f(z)$ fails to be analytic is called a singular point or a singularity of $f(z)$; the Cauchy–Riemann conditions break down at a singularity. Various types of singular points exist.

(1) Isolated singular points: The point $z = z_0$ is called an isolated singular point of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0. If no such δ can be found, we call z_0 a non-isolated singularity.

(2) Poles: If we can find a positive integer n such that $\lim_{z\to z_0}(z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a pole of order n. If $n = 1$, z_0 is called a simple pole. As an example, $f(z) = 1/(z - 2)$ has a simple pole at $z = 2$. But $f(z) = 1/(z - 2)^3$ has a pole of order 3 at $z = 2$.

(3) Branch point: A function has a branch point at z_0 if, upon encircling z_0 and returning to the starting point, the function does not return to the starting value. Thus the function is multiple-valued. An example is $f(z) = \sqrt{z}$, which has a branch point at $z = 0$.

(4) Removable singularities: The singular point z_0 is called a removable singularity of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists. For example, the singular point at $z = 0$ of $f(z) = \sin(z)/z$ is a removable singularity, since $\lim_{z\to 0}\sin(z)/z = 1$.

(5) Essential singularities: A function has an essential singularity at a point z_0 if it has poles of arbitrarily high order which cannot be eliminated by multiplication by $(z - z_0)^n$, which for any finite choice of n. An example is the function $f(z) = e^{1/(z-2)}$, which has an essential singularity at $z = 2$.

(6) Singularities at infinity: The singularity of $f(z)$ at $z = \infty$ is the same type as that of $f(1/w)$ at $w = 0$. For example, $f(z) = z^2$ has a pole of order 2 at $z = \infty$, since $f(1/w) = w^{-2}$ has a pole of order 2 at $w = 0$.

Elementary functions of z

The exponential function e^z *(or* $\exp(z)$)

The exponential function is of fundamental importance, not only for its own sake, but also as a basis for defining all the other elementary functions. In its definition we seek to preserve as many of the characteristic properties of the real exponential function e^x as possible. Specifically, we desire that:

(a) e^z is single-valued and analytic.
(b) $de^z/dz = e^z$.
(c) e^z reduces to e^x when Im $z = 0$.

Recall that if we approach the point z along the x-axis (that is, $\Delta y = 0$, $\Delta x \to 0$), the derivative of an analytic function $f'(z)$ can be written in the form

$$f'(z) = \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

If we let

$$e^z = u + iv,$$

then to satisfy (b) we must have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv.$$

Equating real and imaginary parts gives

$$\frac{\partial u}{\partial x} = u, \tag{6.13}$$

$$\frac{\partial v}{\partial x} = v. \tag{6.14}$$

Eq. (6.13) will be satisfied if we write

$$u = e^x \phi(y), \tag{6.15}$$

where $\phi(y)$ is any function of y. Moreover, since e^z is to be analytic, u and v must satisfy the Cauchy–Riemann equations (6.11). Then using the second of Eqs. (6.11), Eq. (6.14) becomes

$$-\frac{\partial u}{\partial y} = v.$$

Differentiating this with respect to y, we obtain

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y}$$
$$= -\frac{\partial u}{\partial x} \quad \text{(with the aid of the first of Eqs. (6.11))}.$$

Finally, using Eq. (6.13), this becomes

$$\frac{\partial^2 u}{\partial y^2} = -u,$$

which, on substituting Eq. (6.15), becomes

$$e^x \phi''(y) = -e^x \phi(y) \quad \text{or} \quad \phi''(y) = -\phi(y).$$

This is a simple linear differential equation whose solution is of the form

$$\phi(y) = A \cos y + B \sin y.$$

Then

$$u = e^x \phi(y)$$
$$= e^x (A \cos y + B \sin y)$$

and

$$v = -\frac{\partial u}{\partial y} = -e^x(-A \sin y + B \cos y).$$

Therefore

$$e^z = u + iv = e^x[(A \cos y + B \sin y) + i(A \sin y - B \cos y)].$$

If this is to reduce to e^x when $y = 0$, according to (c), we must have

$$e^x = e^x(A - iB)$$

from which we find

$$A = 1 \quad \text{and} \quad B = 0.$$

Finally we find

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y). \tag{6.16}$$

This expression meets our requirements (a), (b), and (c); hence we adopt it as the definition of e^z. It is analytic at each point in the entire z plane, so it is an entire function. Moreover, it satisfies the relation

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}. \tag{6.17}$$

It is important to note that the right hand side of Eq. (6.16) is in standard polar form with the modulus of e^z given by e^x and an argument by y:

$$\text{mod } e^z \equiv |e^z| = e^x \quad \text{and} \quad \arg e^z = y.$$

From Eq. (6.16) we obtain the Euler formula: $e^{iy} = \cos y + i \sin y$. Now let $y = 2\pi$, and since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, the Euler formula gives

$$e^{2\pi i} = 1.$$

Similarly,

$$e^{\pm \pi i} = -1, \quad e^{\pm \pi i/2} = \pm i.$$

Combining this with Eq. (6.17), we find

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z,$$

which shows that e^z is periodic with the imaginary period $2\pi i$. Thus

$$e^{z \pm 2\pi n I} = e^z \quad (n = 0, 1, 2, \ldots). \tag{6.18}$$

Because of the periodicity all the values that $w = f(z) = e^z$ can assume are already assumed in the strip $-\pi < y \leq \pi$. This infinite strip is called the fundamental region of e^z.

Trigonometric and hyperbolic functions

From the Euler formula we obtain

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad (x \text{ real}).$$

This suggests the following definitions for complex z:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \tag{6.19}$$

The other trigonometric functions are defined in the usual way:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z},$$

whenever the denominators are not zero.

From these definitions it is easy to establish the validity of such familiar formulas as:

$$\sin(-z) = -\sin z, \cos(-z) = \cos z, \quad \text{and} \quad \cos^2 z + \sin^2 z = 1,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \quad \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\frac{d \cos z}{dz} = -\sin z, \quad \frac{d \sin z}{dz} = \cos z.$$

Since e^z is analytic for all z, the same is true for the function $\sin z$ and $\cos z$. The functions $\tan z$ and $\sec z$ are analytic except at the points where $\cos z$ is zero, and $\cot z$ and $\operatorname{cosec} z$ are analytic except at the points where $\sin z$ is zero. The

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functions cos z and sec z are even, and the other functions are odd. Since the exponential function is periodic, the trigonometric functions are also periodic, and we have

$$\cos(z \pm 2n\pi) = \cos z, \quad \sin(z \pm 2n\pi) = \sin z,$$

$$\tan(z \pm 2n\pi) = \tan z, \quad \cot(z \pm 2n\pi) = \cot z,$$

where $n = 0, 1, \ldots$.

Another important property also carries over: sin z and cos z have the same zeros as the corresponding real-valued functions:

$$\sin z = 0 \quad \text{if and only if} \quad z = n\pi \quad (n \text{ integer});$$

$$\cos z = 0 \quad \text{if and only if} \quad z = (2n+1)\pi/2 \quad (n \text{ integer}).$$

We can also write these functions in the form $u(x, y) + iv(x, y)$. As an example, we give the details for cos z. From Eq. (6.19) we have

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}(e^{-y}e^{ix} + e^{y}e^{-ix}) \\ &= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)] \\ &= \cos x\frac{e^{y} + e^{-y}}{2} - i\sin x\frac{e^{y} - e^{-y}}{2}\end{aligned}$$

or, using the definitions of the hyperbolic functions of real variables

$$\cos z = \cos(x + iy) = \cos x\cosh y - i\sin x\sinh y;$$

similarly,

$$\sin z = \sin(x + iy) = \sin x\cosh y + i\cos x\sinh y.$$

In particular, taking $x = 0$ in these last two formulas, we find

$$\cos(iy) = \cosh y, \quad \sin(iy) = i\sinh y.$$

There is a big difference between the complex and real sine and cosine functions. The real functions are bounded between -1 and $+1$, but the complex functions can take on arbitrarily large values. For example, if y is real, then cos $iy = \frac{1}{2}(e^{-y} + e^{y}) \to \infty$ as $y \to \infty$ or $y \to -\infty$.

The logarithmic function $w = \ln z$

The real natural logarithm $y = \ln x$ is defined as the inverse of the exponential function $e^{y} = x$. For the complex logarithm, we take the same approach and define $w = \ln z$ which is taken to mean that

$$e^{w} = z \tag{6.20}$$

for each $z \neq 0$.

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Setting $w = u + iv$ and $z = re^{i\theta} = |z|e^{i\theta}$ we have

$$e^{w} = e^{u+iv} = e^{u}e^{iv} = re^{i\theta}.$$

It follows that

$$e^{u} = r = |z| \quad \text{or} \quad u = \ln r = \ln |z|$$

and

$$v = \theta = \arg z.$$

Therefore

$$w = \ln z = \ln r + i\theta = \ln |z| + i \arg z.$$

Since the argument of z is determined only in multiples of 2π, the complex natural logarithm is infinitely many-valued. If we let θ_1 be the principal argument of z, that is, the particular argument of z which lies in the interval $0 \le \theta < 2\pi$, then we can rewrite the last equation in the form

$$\ln z = \ln |z| + i(\theta + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{6.21}$$

For any particular value of n, a unique branch of the function is determined, and the logarithm becomes effectively single-valued. If $n = 0$, the resulting branch of the logarithmic function is called the principal value. Any particular branch of the logarithmic function is analytic, for we have by differentiating the definitive relation $z = e^{w}$,

$$dz/dw = e^{w} = z \quad \text{or} \quad dw/dz = d(\ln z)/dz = 1/z.$$

For a particular value of n the derivative of ln z thus exists for all $z \ne 0$.

For the real logarithm, $y = \ln x$ makes sense when $x > 0$. Now we can take a natural logarithm of a negative number, as shown in the following example.

Example 6.10

$\ln\ -4 = \ln|-4| + i \arg(-4) = \ln 4 + i(\pi + 2n\pi)$; its principal value is $\ln 4 + i(\pi)$, a complex number. This explains why the logarithm of a negative number makes no sense in real variable.

Hyperbolic functions

We conclude this section on "elementary functions" by mentioning briefly the hyperbolic functions; they are defined at points where the denominator does not vanish:

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}), \quad \cosh z = \frac{1}{2}(e^{z} + e^{-z}),$$

$$\tanh z = \sinh z/\cosh z, \quad \coth z = \cosh z/\sinh z,$$

$$\text{sech } z = 1/\cosh z, \quad \text{cosech}\, z = 1/\sinh z.$$

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Since e^z and e^{-z} are entire functions, sinh z and cosh z are also entire functions. The singularities of tanh z and sech z occur at the zeros of cosh z, and the singularities of coth z and cosech z occur at the zeros of sinh z.

As with the trigonometric functions, basic identities and derivative formulas carry over in the same form to the complex hyperbolic functions (just replace x by z). Hence we shall not list them here.

Complex integration

Complex integration is very important. For example, in applications we often encounter real integrals which cannot be evaluated by the usual methods, but we can get help and relief from complex integration. In theory, the method of complex integration yields proofs of some basic properties of analytic functions, which would be very difficult to prove without using complex integration.

The most fundamental result in complex integration is Cauchy's integral theorem, from which the important Cauchy integral formula follows. These will be the subject of this section.

Line integrals in the complex plane

As in real integrals, the indefinite integral $\int f(z)dz$ stands for any function whose derivative is $f(z)$. The definite integral of real calculus is now replaced by integrals of a complex function along a curve. Why? To see this, we can express z in terms of a real parameter t: $z(t) = x(t) + iy(t)$, where, say, $a \le t \le b$. Now as t varies from a to b, the point (x, y) describes a curve in the plane. We say this curve is smooth if there exists a tangent vector at all points on the curve; this means that dx/dt and dy/dt are continuous and do not vanish simultaneously for $a < t < b$.

Let C be such a smooth curve in the complex z plane (Fig. 6.6), and we shall assume that C has a finite length (mathematicians call C a rectifiable curve). Let $f(z)$ be continuous at all points of C. Subdivide C into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and let $a = z_0, b = z_n$. On each arc joining z_{k-1} to z_k $(k = 1, 2, \ldots, n)$ choose a point w_k (possibly $w_k = z_{k-1}$ or $w_k = z_k$) and form the sum

$$S_n = \sum_{k=1}^{n} f(w_k)\Delta z_k \qquad \Delta z_k = z_k - z_{k-1}.$$

Now let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum S_n approaches a limit. If this limit exists and has the same value no matter how the z_js and w_js are chosen, then

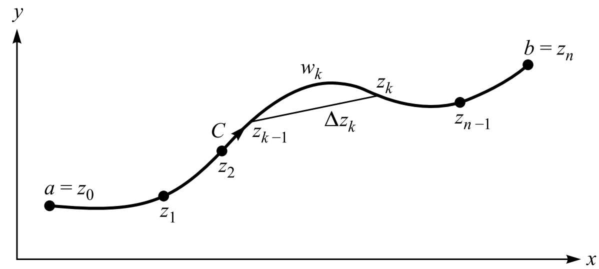

Figure 6.6. Complex line integral.

this limit is called the integral of $f(z)$ along C and is denoted by

$$\int_C f(z)dz \quad \text{or} \quad \int_a^b f(z)dz. \tag{6.22}$$

This is often called a contour integral (with contour C) or a line integral of $f(z)$. Some authors reserve the name contour integral for the special case in which C is a closed curve (so end a and end b coincide), and denote it by the symbol $\oint f(z)dz$.

We now state, without proof, a basic theorem regarding the existence of the contour integral: *If C is piecewise smooth and f(z) is continuous on C, then* $\int_C f(z)dz$ *exists*.

If $f(z) = u(x, y) + iv(x, y)$, the complex line integral can be expressed in terms of real line integrals as

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy), \tag{6.23}$$

where curve C may be open or closed but the direction of integration must be specified in either case. Reversing the direction of integration results in the change of sign of the integral. Complex integrals are, therefore, reducible to curvilinear real integrals and possess the following properties:

(1) $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$;

(2) $\int_C kf(z)dz = k\int_C f(z)dz$, $k =$ any constant (real or complex);

(3) $\int_a^b f(z)dz = -\int_b^a f(z)dz$;

(4) $\int_a^b f(z)dz = \int_a^m f(z)d + \int_m^b f(z)dz$;

(5) $|\int_C f(z)dz| \le ML$, where $M = \max|f(z)|$ on C, and L is the length of C.

Property (5) is very useful, because in working with complex line integrals it is often necessary to establish bounds on their absolute values. We now give a brief proof. Let us go back to the definition:

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta z_k.$$

Now

$$\left| \sum_{k=1}^{n} f(w_k)\Delta z_k \right| \le \sum_{k=1}^{n} |f(w_k)||\Delta z_k| \le M \sum_{k=1}^{n} |\Delta z_k| \le ML,$$

where we have used the fact that $|f(z)| \le M$ for all points z on C and that $\sum |\Delta z_k|$ represents the sum of all the chord lengths joining z_{k-1} and z_k, and that this sum is not greater than the length L of C. Now taking the limit of both sides, and property (5) follows. It is possible to show, more generally, that

$$\left| \int_C f(z)dz \right| \le \int_C |f(z)||dz|. \tag{6.24}$$

Example 6.11
Evaluate the integral $\int_C (z^*)^2 dz$, where C is a straight line joining the points $z = 0$ and $z = 1 + 2i$.

Solution: Since

$$(z^*)^2 = (x - iy)^2 = x^2 - y^2 - 2xyi,$$

we have

$$\int_C (z^*)^2 dz = \int_C [(x^2 - y^2)dx + 2xydy] + i \int_C [-2xydx + (x^2 - y^2)dy].$$

But the Cartesian equation of C is $y = 2x$, and the above integral therefore becomes

$$\int_C (z^*)^2 dz = \int_0^1 5x^2 dx + i \int_0^1 (-10x^2)dx = 5/3 - i10/3.$$

Example 6.12
Evaluate the integral

$$\int_C \frac{dz}{(z - z_0)^{n+1}},$$

where C is a circle of radius r and center at z_0, and n is an integer.

Solution: For convenience, let $z - z_0 = re^{i\theta}$, where θ ranges from 0 to 2π as z ranges around the circle (Fig. 6.7). Then $dz = rie^{i\theta}d\theta$, and the integral becomes

$$\int_0^{2\pi} \frac{rie^{i\theta}d\theta}{r^{n+1}e^{i(n+1)\theta}} = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}d\theta.$$

If $n = 0$, this reduces to

$$i\int_0^{2\pi} d\theta = 2\pi i$$

and if $n \neq 0$, we have

$$\frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)d\theta = 0.$$

This is an important and useful result to which we will refer later.

Cauchy's integral theorem

Cauchy's integral theorem has various theoretical and practical consequences. It states that if $f(z)$ is analytic in a simply-connected region (domain) and on its boundary C, then

$$\oint_C f(z)dz = 0. \tag{6.25}$$

What do we mean by a simply-connected region? A region R (mathematicians prefer the term 'domain') is called simply-connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. That is, a simply-connected region has no hole in it (Fig. 6.7(*a*)); this is not true for a multiply-connected region. The multiply-connected regions of Fig. 6.7(*b*) and (*c*) have respectively one and three holes in them.

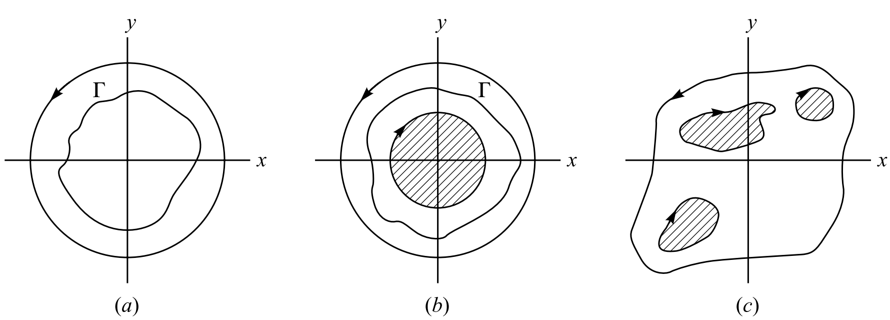

Figure 6.7. Simply-connected and doubly-connected regions.

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Although a rigorous proof of Cauchy's integral theorem is quite demanding and beyond the scope of this book, we shall sketch the main ideas. Note that the integral can be expressed in terms of two-dimensional vector fields **A** and **B**:

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\int_C (vdx + udy)$$

$$= \oint_C \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} + i\oint_C \mathbf{B}(\mathbf{r}) \cdot d\mathbf{r},$$

where

$$\mathbf{A}(\mathbf{r}) = u\hat{e}_1 - v\hat{e}_2, \quad \mathbf{B}(\mathbf{r}) = v\hat{e}_1 + u\hat{e}_2.$$

Applying Stokes' theorem, we obtain

$$\oint_C f(z)dz = \iint_R d\mathbf{a} \cdot (\nabla \times \mathbf{A} + i\nabla \times \mathbf{B})$$

$$= \iint_R dxdy\left[-\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right],$$

where R is the region enclosed by C. Since $f(x)$ satisfies the Cauchy–Riemann conditions, both the real and the imaginary parts of the integral are zero, thus proving Cauchy's integral theorem.

Cauchy's theorem is also valid for multiply-connected regions. For simplicity we consider a doubly-connected region (Fig. 6.8). $f(z)$ is analytic in and on the boundary of the region R between two simple closed curves C_1 and C_2. Construct a cross-cut AF. Then the region bounded by $ABDEAFGHFA$ is simply-connected so by Cauchy's theorem

$$\oint_C f(z)dz = \oint_{ABDEAFGHFA} f(z)dz = 0$$

or

$$\int_{ABDEA} f(z)dz + \int_{AF} f(z)dz + \int_{FGHF} f(z)dz + \int_{FA} f(z)dz = 0.$$

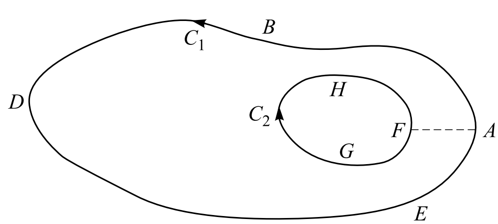

Figure 6.8. Proof of Cauchy's theorem for a doubly-connected region.

But $\int_{AF} f(z)dz = -\int_{FA} f(z)dz$, therefore this becomes

$$\int_{ABDEA} f(z)dzy + \int_{FGHF} f(z)dzy = 0$$

or

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0, \qquad (6.26)$$

where both C_1 and C_2 are traversed in the positive direction (in the sense that an observer walking on the boundary always has the region R on his left). Note that curves C_1 and C_2 are in opposite directions.

If we reverse the direction of C_2 (now C_2 is also counterclockwise, that is, both C_1 and C_2 are in the same direction.), we have

$$\oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0 \quad \text{or} \quad \oint_{C_2} f(z)dz = \oint_{C_1} f(z)dz.$$

Because of Cauchy's theorem, an integration contour can be moved across any region of the complex plane over which the integrand is analytic without changing the value of the integral. It cannot be moved across a hole (the shaded area) or a singularity (the dot), but it can be made to collapse around one, as shown in Fig. 6.9. As a result, an integration contour C enclosing n holes or singularities can be replaced by n separated closed contours C_i, each enclosing a hole or a singularity:

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_i} f(z)dz$$

which is a generalization of Eq. (6.26) to multiply-connected regions.

There is a converse of the Cauchy's theorem, known as Morera's theorem. We now state it without proof:

Morera's theorem:
If $f(z)$ is continuous in a simply-connected region R and the Cauchy's theorem is valid around every simple closed curve C in R, then $f(z)$ is analytic in R.

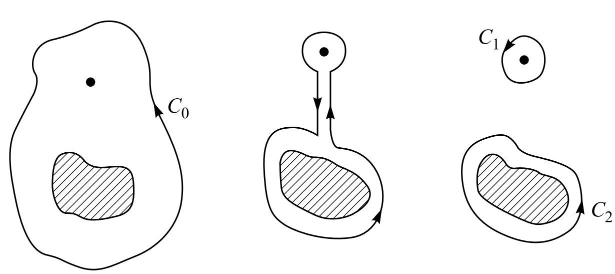

Figure 6.9. Collapsing a contour around a hole and a singularity.

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Example 6.13

Evaluate $\oint_C dz/(z-a)$ where C is any simple closed curve and $z = a$ is (a) outside C, (b) inside C.

Solution: (a) If a is outside C, then $f(z) = 1/(z-a)$ is analytic everywhere inside and on C. Hence by Cauchy's theorem

$$\oint_C dz/(z-a) = 0.$$

(b) If a is inside C and Γ is a circle of radius $\in$ with center at $z = a$ so that Γ is inside C (Fig. 6.10). Then by Eq. (6.26) we have

$$\oint_C dz/(z-a) = \oint_\Gamma dz/(z-a).$$

Now on Γ, $|z-a| = \varepsilon$, or $z - a = \varepsilon e^{i\theta}$, then $dz = i\varepsilon e^{i\theta} d\theta$, and

$$\oint_\Gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i.$$

Cauchy's integral formulas

One of the most important consequences of Cauchy's integral theorem is what is known as Cauchy's integral formula. It may be stated as follows.

> *If $f(z)$ is analytic in a simply-connected region R, and z_0 is any point in the interior of R which is enclosed by a simple closed curve C, then*
>
> $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0} dz, \tag{6.27}$$
>
> *the integration around C being taken in the positive sense (counter-clockwise).*

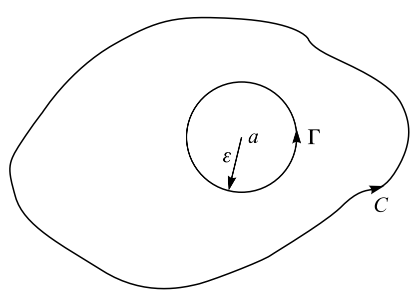

Figure 6.10.

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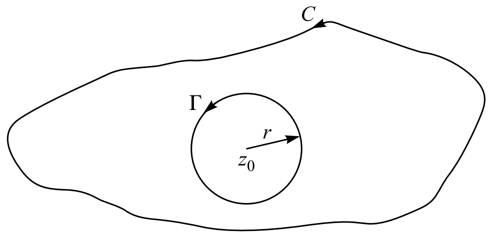

Figure 6.11. Cauchy's integral formula.

To prove this, let Γ be a small circle with center at z_0 and radius r (Fig. 6.11), then by Eq. (6.26) we have

$$\oint_C \frac{f(z)}{z-z_0}dz = \oint_\Gamma \frac{f(z)}{z-z_0}dz.$$

Now $|z-z_0| = r$ or $z - z_0 = re^{i\theta}, 0 \le \theta < 2\pi$. Then $dz = ire^{i\theta}d\theta$ and the integral on the right becomes

$$\oint_\Gamma \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})ire^{i\theta}}{re^{i\theta}}d\theta = i\int_0^{2\pi} f(z_0+re^{i\theta})d\theta.$$

Taking the limit of both sides and making use of the continuity of $f(z)$, we have

$$\oint_C \frac{f(z)}{z-z_0}dz = \lim_{r\to 0}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$$
$$= i\int_0^{2\pi} \lim_{r\to 0} f(z_0+re^{i\theta})d\theta = i\int_0^{2\pi} f(z_0)d\theta = 2\pi i f(z_0),$$

from which we obtain

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \quad \text{q.e.d.}$$

Cauchy's integral formula is also true for multiply-connected regions, but we shall leave its proof as an exercise.

It is useful to write Cauchy's integral formula (6.27) in the form

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z}$$

to emphasize the fact that z can be any point inside the close curve C.

Cauchy's integral formula is very useful in evaluating integrals, as shown in the following example.

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Example 6.14

Evaluate the integral $\oint_C e^z dz/(z^2+1)$, if C is a circle of unit radius with center at (a) $z = i$ and (b) $z = -i$.

Solution: (a) We first rewrite the integral in the form

$$\oint_C \left(\frac{e^z}{z+i}\right)\frac{dz}{z-i},$$

then we see that $f(z) = e^z/(z+i)$ and $z_0 = i$. Moreover, the function $f(z)$ is analytic everywhere within and on the given circle of unit radius around $z = i$. By Cauchy's integral formula we have

$$\oint_C \left(\frac{e^z}{z+i}\right)\frac{dz}{z-i} = 2\pi i f(i) = 2\pi i \frac{e^i}{2i} = \pi(\cos 1 + i \sin 1).$$

(b) We find $z_0 = -i$ and $f(z) = e^z/(z-i)$. Cauchy's integral formula gives

$$\oint_C \left(\frac{e^z}{z-i}\right)\frac{dz}{z+i} = -\pi(\cos 1 - i \sin 1).$$

Cauchy's integral formula for higher derivatives

Using Cauchy's integral formula, we can show that an analytic function $f(z)$ has derivatives of all orders given by the following formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \tag{6.28}$$

where C is any simple closed curve around z_0 and $f(z)$ is analytic on and inside C. Note that this formula implies that each derivative of $f(z)$ is itself analytic, since it possesses a derivative.

We now prove the formula (6.28) by induction on n. That is, we first prove the formula for $n = 1$:

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^2}.$$

As shown in Fig. 6.12, both z_0 and $z_0 + h$ lie in R, and

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}.$$

Using Cauchy's integral formula we obtain

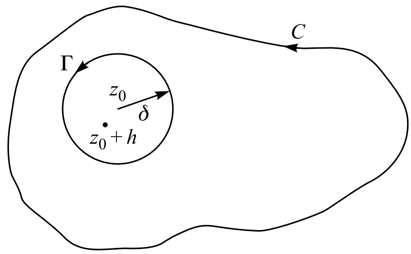

Figure 6.12.

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$
$$= \lim_{h\to 0} \frac{1}{2\pi i h} \oint_C \left\{ \frac{1}{z-(z_0+h)} - \frac{1}{z-z_0} \right\} f(z)dz.$$

Now

$$\frac{1}{h}\left[\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right] = \frac{1}{(z-z_0)^2} + \frac{h}{(z-z_0-h)(z-z_0)^2}.$$

Thus,

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz + \frac{1}{2\pi i}\lim_{h\to 0} h \oint_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz.$$

The proof follows if the limit on the right hand side approaches zero as $h \to 0$. To show this, let us draw a small circle Γ of radius δ centered at z_0 (Fig. 6.12), then

$$\frac{1}{2\pi i}\lim_{h\to 0} h \oint_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz = \frac{1}{2\pi i}\lim_{h\to 0} h \oint_\Gamma \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz.$$

Now choose h so small (in absolute value) that $z_0 + h$ lies in Γ and $|h| < \delta/2$, and the equation for Γ is $|z - z_0| = \delta$. Thus, we have $|z - z_0 - h| \geq |z - z_0| - |h| > \delta - \delta/2 = \delta/2$. Next, as $f(z)$ is analytic in R, we can find a positive number M such that $|f(z)| \leq M$. And the length of Γ is $2\pi\delta$. Thus,

$$\left| \frac{h}{2\pi i}\oint_\Gamma \frac{f(z)dz}{(z-z_0-h)(z-z_0)^2} \right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\delta)}{(\delta/2)(\delta^2)} = \frac{2|h|M}{\delta^2} \to 0 \quad \text{as} \quad h \to 0,$$

proving the formula for $f'(z_0)$.

For $n = 2$, we begin with

$$\frac{f'(z_0+h)-f'(z_0)}{h} = \frac{1}{2\pi ih}\oint_C \left\{\frac{1}{(z-z_0h)^2} - \frac{1}{(z-z_0)^2}\right\} f(z)dz$$
$$= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz + \frac{h}{2\pi i}\oint_C \frac{3(z-z_0)-2h}{(z-z_0-h)^2(z-z_0)^3} f(z)dz.$$

The result follows on taking the limit as $h \to 0$ if the last term approaches zero. The proof is similar to that for the case $n = 1$, for using the fact that the integral around C equals the integral around Γ, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{3(z-z_0)-2h}{(z-z_0-h)^2(z-z_0)^3} f(z)dz\right| \le \frac{|h|}{2\pi}\frac{M(2\pi\delta)}{(\delta/2)^2\delta^3} = \frac{4|h|M}{\delta^4},$$

assuming M exists such that $|[3(z-z_0)-2h]f(z)| < M$.

In a similar manner we can establish the results for $n = 3, 4, \ldots$. We leave it to the reader to complete the proof by establishing the formula for $f^{(n+1)}(z_0)$, assuming that $f^{(n)}(z_0)$ is true.

Sometimes Cauchy's integral formula for higher derivatives can be used to evaluate integrals, as illustrated by the following example.

Example 6.15
Evaluate

$$\oint_C \frac{e^{2z}}{(z+1)^4}dz,$$

where C is any simple closed path not passing through -1. Consider two cases:

(*a*) C does not enclose -1. Then $e^{2z}/(z+1)^4$ is analytic on and inside C, and the integral is zero by Cauchy's integral theorem.
(*b*) C encloses -1. Now Cauchy's integral formula for higher derivatives applies.

Solution: Let $f(z) = e^{2z}$, then

$$f^{(3)}(-1) = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}dz.$$

Now $f^{(3)}(-1) = 8e^{-2}$, hence

$$\oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}f^{(3)}(-1) = \frac{8\pi}{3}e^{-2}i.$$

Series representations of analytic functions

We now turn to a very important notion: series representations of analytic functions. As a prelude we must discuss the notion of convergence of complex series. Most of the definitions and theorems relating to infinite series of real terms can be applied with little or no change to series whose terms are complex.

Complex sequences

A complex sequence is an ordered list which assigns to each positive integer n a complex number z_n:

$$z_1, z_2, \ldots, z_n, \ldots.$$

The numbers z_n are called the terms of the sequence. For example, both $i, i^2, \ldots, i^n, \ldots$ or $1 + i, (1 + i)/2, (1 + i)/4, (1 + i)/8, \ldots$ are complex sequences. The nth term of the second sequence is $(1 + i)/2^{n-1}$. A sequence $z_1, z_2, \ldots, z_n, \ldots$ is said to be convergent with the limit l (or simply to converge to the number l) if, given $\varepsilon > 0$, we can find a positive integer N such that $|z_n - l| < \varepsilon$ for each $n \geq N$ (Fig. 6.13). Then we write

$$\lim_{n\to\infty} z_n = l.$$

In words, or geometrically, this means that each term z_n with $n > N$ (that is, $z_N, z_{N+1}, z_{N+2}, \ldots$) lies in the open circular region of radius ε with center at l. In general, N depends on the choice of ε. Here is an illustrative example.

Example 6.17
Using the definition, show that $\lim_{n\to\infty}(1 + z/n) = 1$ for all z.

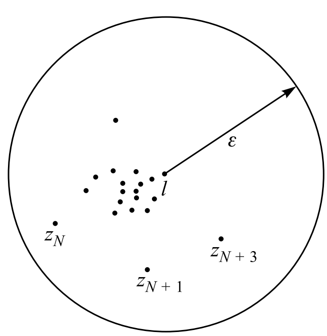

Figure 6.13. Convergent complex sequence.

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Solution: Given any number $\varepsilon > 0$, we must find N such that

$$\left|1 + \frac{z}{n} - 1\right| < \varepsilon, \quad \text{for all} \quad n > N$$

from which we find

$$|z/n| < \varepsilon$$

or

$$|z|/n < \varepsilon \quad \text{if} \quad n > |z|/\varepsilon \equiv N.$$

Setting $z_n = x_n + iy_n$, we may consider a complex sequence $z_1, z_2, \ldots, z_n$ in terms of real sequences, the sequence of the real parts and the sequence of the imaginary parts: $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$. If the sequence of the real parts converges to the number A, and the sequence of the imaginary parts converges to the number B, then the complex sequence $z_1, z_2, \ldots, z_n$ converges to the limit $A + iB$, as illustrated by the following example.

Example 6.18

Consider the complex sequence whose nth term is

$$z_n = \frac{n^2 - 2n + 3}{3n^2 - 4} + i\frac{2n - 1}{2n + 1}.$$

Setting $z_n = x_n + iy_n$, we find

$$x_n = \frac{n^2 - 2n + 3}{3n^2 - 4} = \frac{1 - (2/n) + (3/n^2)}{3 - 4/n^2} \quad \text{and} \quad y_n = \frac{2n - 1}{2n + 1} = \frac{2 - 1/n}{2 + 1/n}.$$

As $n \to \infty, x_n \to 1/3$ and $y_n \to 1$, thus, $z_n \to 1/3 + i$.

Complex series

We are interested in complex series whose terms are complex functions

$$f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots. \tag{6.29}$$

The sum of the first n terms is

$$S_n(z) = f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z),$$

which is called the nth partial sum of the series (6.29). The sum of the remaining terms after the nth term is called the remainder of the series.

We can now associate with the series (6.29) the sequence of its partial sums $S_1, S_2, \ldots$. If this sequence of partial sums is convergent, then the series converges; and if the sequence diverges, then the series diverges. We can put this in a formal way. The series (6.29) is said to converge to the sum $S(z)$ in a region R if for any

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$\varepsilon > 0$ there exists an integer N depending in general on ε and on the particular value of z under consideration such that

$$|S_n(z) - S(z)| < \varepsilon \quad \text{for all } n > N$$

and we write

$$\lim_{n\to\infty} S_n(z) = S(z).$$

The difference $S_n(z) - S(z)$ is just the remainder after n terms, $R_n(z)$; thus the definition of convergence requires that $|R_n(z)| \to 0$ as $n \to \infty$.

If the absolute values of the terms in (6.29) form a convergent series

$$|f_1(z)| + |f_2(z)| + |f_3(z)| + \cdots + |f_n(z)| + \cdots$$

then series (6.29) is said to be absolutely convergent. If series (6.29) converges but is not absolutely convergent, it is said to be conditionally convergent. The terms of an absolutely convergent series can be rearranged in any manner whatsoever without affecting the sum of the series whereas rearranging the terms of a conditionally convergent series may alter the sum of the series or even cause the series to diverge.

As with complex sequences, questions about complex series can also be reduced to questions about real series, the series of the real part and the series of the imaginary part. From the definition of convergence it is not difficult to prove the following theorem:

A necessary and sufficient condition that the series of complex terms

$$f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots$$

should convergence is that the series of the real parts and the series of the imaginary parts of these terms should each converge. Moreover, if

$$\sum_{n=1}^{\infty} \operatorname{Re} f_n \quad \text{and} \quad \sum_{n=1}^{\infty} \operatorname{Im} f_n$$

converge to the respective functions $R(z)$ and $I(z)$, then the given series converges to $R(z) + I(z)$, and the series $f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots$ converges to $R(z) + iI(z)$.

Of all the tests for the convergence of infinite series, the most useful is probably the familiar *ratio test*, which applies to real series as well as complex series.

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Ratio test

Given the series $f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots$, *the series converges absolutely if*

$$0 < |r(z)| = \lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| < 1 \tag{6.30}$$

and diverges if $|r(z)| > 1$. *When* $|r(z)| = 1$, *the ratio test provides no information about the convergence or divergence of the series.*

Example 6.19

Consider the complex series

$$\sum_n S_n = \sum_{n=0}^{\infty}(2^{-n} + ie^{-n}) = \sum_{n=0}^{\infty} 2^{-n} + i\sum_{n=0}^{\infty} e^{-n}.$$

The ratio tests on the real and imaginary parts show that both converge:

$$\lim_{n\to\infty}\left|\frac{2^{-(n+1)}}{2^{-n}}\right| = \frac{1}{2}, \text{ which is positive and less than 1;}$$

$$\lim_{n\to\infty}\left|\frac{e^{-(n+1)}}{e^{-n}}\right| = \frac{1}{e}, \text{ which is also positive and less than 1.}$$

One can prove that the full series converges to

$$\sum_{n=1}^{\infty} S_n = \frac{1}{1-1/2} + i\frac{1}{1-e^{-1}}.$$

Uniform convergence and the Weierstrass M-test

To establish conditions, under which series can legitimately be integrated or differentiated term by term, the concept of uniform convergence is required:

> *A series of functions is said to converge uniformly to the function* $S(z)$ *in a region* R, *either open or closed, if corresponding to an arbitrary* $\varepsilon < 0$ *there exists an integral* N, *depending on* ε *but not on* z, *such that for every value of* z *in* R
>
> $$|S(z) - S_n(z)| < \varepsilon \quad \textit{for all } n > N.$$

One of the tests for uniform convergence is the Weierstrass M-test (a sufficient test).

If a sequence of positive constants $\{M_n\}$ exists such that $|f_n(z)| \leq M_n$ for all positive integers n and for all values of z in a given region R, and if the series

$$M_1 + M_2 + \cdots + M_n + \cdots$$

is convergent, then the series

$$f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots$$

converges uniformly in R.

As an illustrative example, we use it to test for uniform convergence of the series

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$$

in the region $|z| \leq 1$. Now

$$|u_n| = \frac{|z|^n}{n\sqrt{n+1}} \leq \frac{1}{n^{3/2}}$$

if $|z| \leq 1$. Calling $M_n = 1/n^{3/2}$, we see that $\sum M_n$ converges, as it is a p series with $p = 3/2$. Hence by Wierstrass M-test the given series converges uniformly (and absolutely) in the indicated region $|z| \leq 1$.

Power series and Taylor series

Power series are one of the most important tools of complex analysis, as power series with non-zero radii of convergence represent analytic functions. As an example, the power series

$$S = \sum_{n=0}^{\infty} a_n z^n \tag{6.31}$$

clearly defines an analytic function as long as the series converge. We will only be interested in absolute convergence. Thus we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| < 1 \quad \text{or} \quad |z| < R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

where R is the radius of convergence since the series converges for all z lying strictly inside a circle of radius R centered at the origin. Similarly, the series

$$S = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges within a circle of radius R centered at z_0.

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Notice that the Eq. (6.31) is just a Taylor series at the origin of a function with $f^n(0) = a_n n!$. Every choice we make for the infinite variables a_n defines a new function with its own set of derivatives at the origin. Of course we can go beyond the origin, and expand a function in a Taylor series centered at $z = z_0$. Thus in the complex analysis there is a Taylor expansion for every analytic function. This is the question addressed by *Taylor's theorem* (named after the English mathematician Brook Taylor, 1685–1731):

If $f(z)$ is analytic throughout a region R bounded by a simple closed curve C, and if z and a are both interior to C, then $f(z)$ can be expanded in a Taylor series centered at $z = a$ for $|z - a| < R$:

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + f^n(a)\frac{(z-a)^{n-1}}{n!} + R_n, \tag{6.32}$$

where the remainder R_n is given by

$$R_n(z) = (z-a)^n \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^n(w-z)}.$$

Proof: To prove this, we first rewrite Cauchy's integral formula as

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}\left[\frac{1}{1-(z-a)/(w-a)}\right]dw. \tag{6.33}$$

For later use we note that since w is on C while z is inside C,

$$\left|\frac{z-a}{w-a}\right| < 1.$$

From the geometric progression

$$1 + q + q^2 + \cdots + q^n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$

we obtain the relation

$$\frac{1}{1-q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1-q}.$$

By setting $q = (z - a)/(w - a)$ we find

$$\frac{1}{1 - [(z-a)/(w-a)]} = 1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \frac{[(z-a)/(w-a)]^{n+1}}{(w-z)/(w-a)}.$$

We insert this into Eq. (6.33). Since z and a are constant, we may take the powers of $(z - a)$ out from under the integral sign, and then Eq. (6.33) takes the form

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{w-a} + \frac{z-a}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^2} + \cdots + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^{n+1}} + R_n(z).$$

Using Eq. (6.28), we may write this expansion in the form

$$f(z) = f(a) + \frac{z-a}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + R_n(z),$$

where

$$R_n(z) = (z-a)^n \frac{1}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^n(w-z)}.$$

Clearly, the expansion will converge and represent $f(z)$ if and only if $\lim_{n\to\infty} R_n(z) = 0$. This is easy to prove. Note that w is on C while z is inside C, so we have $|w - z| > 0$. Now $f(z)$ is analytic inside C and on C, so it follows that the absolute value of $f(w)/(w - z)$ is bounded, say,

$$\left|\frac{f(w)}{w-z}\right| < M$$

for all w on C. Let r be the radius of C, then $|w - a| = r$ for all w on C, and C has the length $2\pi r$. Hence we obtain

$$|R_n| = \frac{|z-a|^n}{2\pi}\left|\oint_C \frac{f(w)dw}{(w-a)^n(w-z)}\right| < \frac{|z-a|^n}{2\pi} M \frac{1}{r^n} 2\pi r$$
$$= Mr\left|\frac{z-a}{r}\right|^n \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$f(z) = f(a) + \frac{z-a}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a)$$

is a valid representation of $f(z)$ at all points in the interior of any circle with its center at a and within which $f(z)$ is analytic. This is called the Taylor series of $f(z)$ with center at a. And the particular case where $a = 0$ is called the Maclaurin series of $f(z)$ [Colin Maclaurin 1698–1746, Scots mathematician].

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The Taylor series of $f(z)$ converges to $f(z)$ only within a circular region around the point $z = a$, the circle of convergence; and it diverges everywhere outside this circle.

Taylor series of elementary functions

Taylor series of analytic functions are quite similar to the familiar Taylor series of real functions. Replacing the real variable in the latter series by a complex variable we may 'continue' real functions analytically to the complex domain. The following is a list of Taylor series of elementary functions: in the case of multiple-valued functions, the principal branch is used.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots, \qquad |z| < \infty,$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots, \qquad |z| < \infty,$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots, \qquad |z| < \infty,$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \qquad |z| < \infty,$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \qquad |z| < \infty,$$

$$\ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots, \qquad |z| < 1.$$

Example 6.20
Expand $(1 - z)^{-1}$ about a.

Solution:

$$\frac{1}{1-z} = \frac{1}{(1-a)-(z-a)} = \frac{1}{1-a}\frac{1}{1-(z-a)/(1-a)} = \frac{1}{1-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{1-a}\right)^n.$$

We have established two surprising properties of complex analytic functions:

(1) *They have derivatives of all order.*
(2) *They can always be represented by Taylor series.*

This is not true in general for real functions; there are real functions which have derivatives of all orders but cannot be represented by a power series.

Example 6.21

Expand $\ln(a+z)$ about a.

Solution: Suppose we know the Maclaurin series, then

$$\ln(1+z) = \ln(1+a+z-a) = \ln(1+a)\left(1+\frac{z-a}{1+a}\right) = \ln(1+a) + \ln\left(1+\frac{z-a}{1+a}\right)$$

$$= \ln(1+a) + \left(\frac{z-a}{1+a}\right) - \frac{1}{2}\left(\frac{z-a}{1+a}\right)^2 + \frac{1}{3}\left(\frac{z-a}{1+a}\right)^3 - + \cdots.$$

Example 6.22

Let $f(z) = \ln(1+z)$, and consider that branch which has the value zero when $z = 0$.

(a) Expand $f(z)$ in a Taylor series about $z = 0$, and determine the region of convergence.

(b) Expand $\ln[(1+z)/(1-z)]$ in a Taylor series about $z = 0$.

Solution: (a)

$$\begin{aligned}
f(z) &= \ln(1+z) & f(0) &= 0 \\
f'(z) &= (1+z)^{-1} & f'(0) &= 1 \\
f''(z) &= -(1+z)^{-2} & f''(0) &= -1 \\
f'''(z) &= 2(1+z)^{-3} & f'''(0) &= 2! \\
&\vdots & &\vdots \\
f^{(n+1)}(z) &= (-1)^n n!(1+n)^{(n+1)} & f^{(n+1)}(0) &= (-1)^n n!.
\end{aligned}$$

Then

$$f(z) = \ln(1+z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - \cdots.$$

The nth term is $u_n = (-1)^{n-1} z^n/n$. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

and the series converges for $|z| < 1$.

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(*b*) $\ln[(1+z)/(1-z)] = \ln(1+z) - \ln(1-z)$. Next, replacing z by $-z$ in Taylor's expansion for $\ln(1+z)$, we have

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots.$$

Then by subtraction, we obtain

$$\ln\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}.$$

Laurent series

In many applications it is necessary to expand a function $f(z)$ around points where or in the neighborhood of which the function is not analytic. The Taylor series is not applicable in such cases. A new type of series known as the Laurent series is required. The following is a representation which is valid in an annular ring bounded by two concentric circles of C_1 and C_2 such that $f(z)$ is single-valued and analytic in the annulus and at each point of C_1 and C_2, see Fig. 6.14. The function $f(z)$ may have singular points outside C_1 and inside C_2. Hermann Laurent (1841–1908, French mathematician) proved that, at any point in the annular ring bounded by the circles, $f(z)$ can be represented by the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \tag{6.34}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)dw}{(w-a)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{6.35}$$

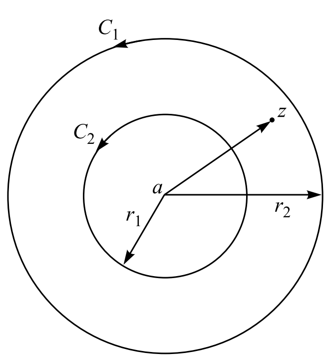

Figure 6.14. Laurent theorem.

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each integral being taken in the counterclockwise sense around curve C lying in the annular ring and encircling its inner boundary (that is, C is any concentric circle between C_1 and C_2).

To prove this, let z be an arbitrary point of the annular ring. Then by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{w-z} + \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{w-z},$$

where C_2 is traversed in the counterclockwise direction and C_2 is traversed in the clockwise direction, in order that the entire integration is in the positive direction. Reversing the sign of the integral around C_2 and also changing the direction of integration from clockwise to counterclockwise, we obtain

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{w-z}.$$

Now

$$1/(w-z) = [1/(w-a)]\{1/[1-(z-a)/(w-a)]\},$$

$$-1/(w-z) = 1/(z-w) = [1/(z-a)]\{1/[1-(w-a)/(z-a)]\}.$$

Substituting these into $f(z)$ we obtain:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{w-z} \\ &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\left[\frac{1}{1-(z-a)/(w-a)}\right]dw \\ &\quad + \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-a}\left[\frac{1}{1-(w-a)/(z-a)}\right]dw. \end{aligned}$$

Now in each of these integrals we apply the identity

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} + \frac{q^n}{1-q}$$

to the last factor. Then

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\left[1+\frac{z-a}{w-a}+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\frac{(z-a)^n/(w-a)^n}{1-(z-a)/(w-a)}\right]dw$$

$$+\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-a}\left[1+\frac{w-a}{z-a}+\cdots+\left(\frac{w-a}{z-a}\right)^{n-1}+\frac{(w-a)^n/(z-a)^n}{1-(w-a)/(z-a)}\right]dw$$

$$=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)dw}{w-a}+\frac{z-a}{2\pi i}\oint_{C_2}\frac{f(w)dw}{(w-a)^2}+\cdots+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_2}\frac{f(w)dw}{(w-a)^n}+R_{n1}$$

$$+\frac{1}{2\pi i(z-a)}\oint_{C_2} f(w)dw+\frac{1}{2\pi i(z-a)^2}\oint_{C_1}(w-a)f(w)dw+\cdots$$

$$+\frac{1}{2\pi i(z-a)^n}\oint_{C_1}(w-a)^{n-1}f(w)dw+R_{n2},$$

where

$$R_{n1} = \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)dw}{(w-a)^n(w-z)},$$

$$R_{n2} = \frac{1}{2\pi i(z-a)^n}\oint_{C_2}\frac{(w-a)^n f(w)dw}{z-w}.$$

The theorem will be established if we can show that $\lim_{n\to\infty} R_{n2} = 0$ and $\lim_{n\to\infty} R_{n1} = 0$. The proof of $\lim_{n\to\infty} R_{n1} = 0$ has already been given in the derivation of the Taylor series. To prove the second limit, we note that for values of w on C_2

$$|w-a| = r_1, |z-a| = \rho \quad \text{say,} \; |z-w| = |(z-a)-(w-a)| \geq \rho - r_1,$$

and

$$|(f(w)| \leq M,$$

where M is the maximum of $|f(w)|$ on C_2. Thus

$$|R_{n_2}| = \left|\frac{1}{2\pi i(z-a)^n}\oint_{C_2}\frac{(w-a)^n f(w)dw}{z-w}\right| \leq \frac{1}{|2\pi i||z-a|^n}\oint_{C_2}\frac{|w-a|^n|f(w)||dw|}{|z-w|}$$

or

$$|R_{n_2}| \leq \frac{r_1^n M}{2\pi\rho^n(\rho-r_1)}\oint_{C_2}|dw| = \frac{M}{2\pi}\left(\frac{r_1}{\rho}\right)^n\frac{2\pi r_1}{\rho-r_1}.$$

Since $r_1/\rho < 1$, the last expression approaches zero as $n \to \infty$. Hence $\lim_{n\to\infty} R_{n2} = 0$ and we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)dw}{w-a} + \left[\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)dw}{(w-a)^2}\right](z-a)$$
$$+ \left[\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)dw}{(w-a)^3}\right](z-a)^2 + \cdots$$
$$+ \left[\frac{1}{2\pi i}\oint_{C_2} f(w)dw\right]\frac{1}{z-a} + \left[\frac{1}{2\pi i}\oint_{C_2} (w-a)f(w)dw\right]\frac{1}{(z-a)^2} + \cdots.$$

Since $f(z)$ is analytic throughout the region between C_1 and C_2, the paths of integration C_1 and C_2 can be replaced by any other curve C within this region and enclosing C_2. And the resulting integrals are precisely the coefficients a_n given by Eq. (6.35). This proves the Laurent theorem.

It should be noted that the coefficients of the positive powers $(z-a)$ in the Laurent expansion, while identical in form with the integrals of Eq. (6.28), cannot be replaced by the derivative expressions

$$\frac{f^n(a)}{n!}$$

as they were in the derivation of Taylor series, since $f(z)$ is not analytic throughout the entire interior of C_2 (or C), and hence Cauchy's generalized integral formula cannot be applied.

In many instances the Laurent expansion of a function is not found through the use of the formula (6.34), but rather by algebraic manipulations suggested by the nature of the function. In particular, in dealing with quotients of polynomials it is often advantageous to express them in terms of partial fractions and then expand the various denominators in series of the appropriate form through the use of the binomial expansion, which we assume the reader is familiar with:

$$(s+t)^n = s^n + ns^{n-1}t\frac{n(n-1)}{2!}s^{n-2}t^2 + \frac{n(n-1)(n-2)}{3!}s^{n-3}t^3 + \cdots.$$

This expansion is valid for all values of n if $|s| > |t|$. *If* $|s| \le |t|$ *the expansion is valid only if n is a non-negative integer.*

That such procedures are correct follows from the fact that *the Laurent expansion of a function over a given annular ring is unique*. That is, if an expansion of the Laurent type is found by any process, it must be the Laurent expansion.

Example 6.23

Find the Laurent expansion of the function $f(z) = (7z-2)/[(z+1)z(z-2)]$ in the annulus $1 < |z+1| < 3$.

Solution: We first apply the method of partial fractions to $f(z)$ and obtain

$$f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2}.$$

Now the center of the given annulus is $z = -1$, so the series we are seeking must be one involving powers of $z + 1$. This means that we have to modify the second and third terms in the partial fraction representation of $f(z)$:

$$f(z) = \frac{-3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3},$$

but the series for $[(z+1)-3]^{-1}$ converges only where $|z+1| > 3$, whereas we require an expansion valid for $|z+1| < 3$. Hence we rewrite the third term in the other order:

$$\begin{aligned} f(z) &= \frac{-3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{-3+(z+1)} \\ &= -3(z+1)^{-1} + [(z+1)-1]^{-1} + 2[-3+(z+1)]^{-1} \\ &= \cdots + (z+1)^{-2} - 2(z+1)^{-1} - \frac{2}{3} - \frac{2}{9}(z+1) \\ &\quad - \frac{2}{27}(z+1)^2 - \cdots, \qquad 1 < |z+1| < 3. \end{aligned}$$

Example 6.24

Given the following two functions:

$$(a)\ e^{3z}(z+1)^{-3}, \qquad (b)\ (z+2)\sin\frac{1}{z+2},$$

find Laurent series about the singularity for each of the functions, name the singularity, and give the region of convergence.

Solution: (*a*) $z = -1$ is a triple pole (pole of order 3). Let $z + 1 = u$, then $z = u - 1$ and

$$\begin{aligned} \frac{e^{3z}}{(z+1)^3} &= \frac{e^{3(u-1)}}{u^3} = e^{-3}\frac{e^{3u}}{u^3} = \frac{e^{-3}}{u^3}\left(1 + 3u + \frac{(3u)^2}{2!} + \frac{(3u)^3}{3!} + \frac{(3u)^4}{4!} + \cdots\right) \\ &= e^{-3}\left(\frac{1}{(z+1)^3} + \frac{3}{(z+1)^2} + \frac{9}{2(z+1)} + \frac{9}{2} + \frac{27(z+1)}{8} + \cdots\right). \end{aligned}$$

The series converges for all values of $z \neq -1$.

(*b*) $z = -2$ is an essential singularity. Let $z + 2 = u$, then $z = u - 2$, and

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$$(z+2)\sin\frac{1}{z+2} = u\sin\frac{1}{u} = u\left(\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} + \cdots\right)$$
$$= 1 - \frac{1}{6(z+2)^2} + \frac{1}{120(z+2)^4} - + \cdots.$$

The series converges for all values of $z \neq -2$.

Integration by the method of residues

We now turn to integration by the method of residues which is useful in evaluating both real and complex integrals. We first discuss briefly the theory of residues, then apply it to evaluate certain types of real definite integrals occurring in physics and engineering.

Residues

If $f(z)$ is single-valued and analytic in a neighborhood of a point $z = a$, then, by Cauchy's integral theorem,

$$\oint_C f(z)dz = 0$$

for any contour in that neighborhood. But if $f(z)$ has a pole or an isolated essential singularity at $z = a$ and lies in the interior of C, then the above integral will, in general, be different from zero. In this case we may represent $f(z)$ by a Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots,$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

The sum of all the terms containing negative powers, namely $a_{-1}/(z-a) + a_{-2}/(z-a)^2 + \cdots$, is called the principal part of $f(z)$ at $z = a$. In the special case $n = -1$, we have

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)dz$$

or

$$\oint_C f(z)dz = 2\pi i a_{-1}, \qquad (6.36)$$

the integration being taken in the counterclockwise sense around a simple closed curve C that lies in the region $0 < |z - a| < D$ and contains the point $z = a$, where D is the distance from a to the nearest singular point of $f(z)$. The coefficient a_{-1} is called the residue of $f(z)$ at $z = a$, and we shall use the notation

$$a_{-1} = \operatorname*{Res}_{z=a} f(z). \tag{6.37}$$

We have seen that Laurent expansions can be obtained by various methods, without using the integral formulas for the coefficients. Hence, we may determine the residue by one of those methods and then use the formula (6.36) to evaluate contour integrals. To illustrate this, let us consider the following simple example.

Example 6.25
Integrate the function $f(z) = z^{-4} \sin z$ around the unit circle C in the counterclockwise sense.

Solution: Using

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots,$$

we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots.$$

We see that $f(z)$ has a pole of third order at $z = 0$, the corresponding residue is $a_{-1} = -1/3!$, and from Eq. (6.36) it follows that

$$\oint \frac{\sin z}{z^4} dz = 2\pi i a_{-1} = -i\frac{\pi}{3}.$$

There is a simple standard method for determining the residue in the case of a pole. If $f(z)$ has a simple pole at a point $z = a$, the corresponding Laurent series is of the form

$$f(z) = \sum_{n=-1}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a},$$

where $a_{-1} \neq 0$. Multiplying both sides by $z - a$, we have

$$(z-a)f(z) = (z-a)[a_0 + a_1(z-a) + \cdots] + a_{-1}$$

and from this we have

$$\operatorname*{Res}_{z=a} f(z) = a_{-1} = \lim_{z \to a} (z-a)f(z). \tag{6.38}$$

Another useful formula is obtained as follows. If $f(z)$ can be put in the form

$$f(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are analytic at $z = a$, $p(z) \neq 0$, and $q(z) = 0$ at $z = a$ (that is, $q(z)$ has a simple zero at $z = a$). Consequently, $q(z)$ can be expanded in a Taylor series of the form

$$q(z) = (z-a)q'(a) + \frac{(z-a)^2}{2!}q''(a) + \cdots.$$

Hence

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)\frac{p(z)}{q(z)} = \lim_{z\to a}\frac{(z-a)p(z)}{(z-a)[q'(a) + (z-a)q''(a)/2 + \cdots]} = \frac{p(a)}{q'(a)}. \tag{6.39}$$

Example 6.26

The function $f(z) = (4-3z)/(z^2-z)$ is analytic except at $z = 0$ and $z = 1$ where it has simple poles. Find the residues at these poles.

Solution: We have $p(z) = 4-3z, q(z) = z^2 - z$. Then from Eq. (6.39) we obtain

$$\operatorname*{Res}_{z=0} f(z) = \left[\frac{4-3z}{2z-1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{4-3z}{2z-1}\right]_{z=1} = 1.$$

We now consider poles of higher orders. If $f(z)$ has a pole of order $m > 1$ at a point $z = a$, the corresponding Laurent series is of the form

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-m}}{(z-a)^m},$$

where $a_{-m} \neq 0$ and the series converges in some neighborhood of $z = a$, except at the point itself. By multiplying both sides by $(z-a)^m$ we obtain

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + a_{-m+2}(z-a)^2 + \cdots + a_{-m+(m-1)}(z-a)^{(m-1)} + (z-a)^m[a_0 + a_1(z-a) + \cdots].$$

This represents the Taylor series about $z = a$ of the analytic function on the left hand side. Differentiating both sides $(m-1)$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] = (m-1)!a_{-1} + m(m-1)\cdots 2a_0(z-a) + \cdots.$$

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Thus on letting $z \to a$

$$\lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] = (m-1)!a_{-1},$$

that is,

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \left\{ \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \right\}. \tag{6.40}$$

Of course, in the case of a rational function $f(z)$ the residues can also be determined from the representation of $f(z)$ in terms of partial fractions.

The residue theorem

So far we have employed the residue method to evaluate contour integrals whose integrands have only a single singularity inside the contour of integration. Now consider a simple closed curve C containing in its interior a number of isolated singularities of a function $f(z)$. If around each singular point we draw a circle so small that it encloses no other singular points (Fig. 6.15), these small circles, together with the curve C, form the boundary of a multiply-connected region in which $f(z)$ is everywhere analytic and to which Cauchy's theorem can therefore be applied. This gives

$$\frac{1}{2\pi i}\left[\oint_C f(z)dz + \oint_{C_1} f(z)dz + \cdots + \oint_{C_m} f(z)dz\right] = 0.$$

If we reverse the direction of integration around each of the circles and change the sign of each integral to compensate, this can be written

$$\frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_{C_1} f(z)dz + \frac{1}{2\pi i}\oint_{C_2} f(z)dz + \cdots + \frac{1}{2\pi i}\oint_{C_m} f(z)dz,$$

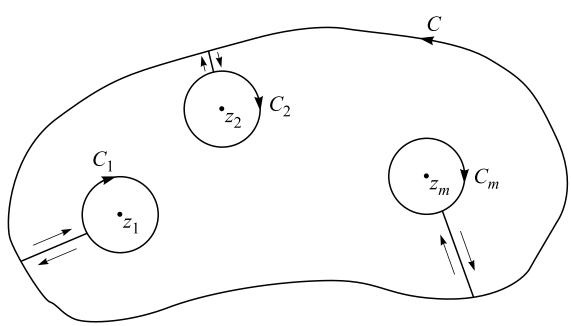

Figure 6.15. Residue theorem.

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where all the integrals are now to be taken in the counterclockwise sense. But the integrals on the right are, by definition, just the residues of $f(z)$ at the various isolated singularities within C. Hence we have established an important theorem, the residue theorem:

> *If $f(z)$ is analytic inside a simple closed curve C and on C, except at a finite number of singular points $a_1, a_2, \ldots, a_m$ in the interior of C, then*
>
> $$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=a_j} f(z) = 2\pi i\,(r_1 + r_2 + \cdots + r_m), \qquad (6.41)$$
>
> *where r_j is the residue of $f(z)$ at the singular point a_j.*

Example 6.27
The function $f(z) = (4-3z)/(z^2-z)$ has simple poles at $z=0$ and $z=1$; the residues are -4 and 1, respectively (cf. Example 6.26). Therefore

$$\oint_C \frac{4-3z}{z^2-z}dz = 2\pi i(-4+1) = -6\pi i$$

for every simple closed curve C which encloses the points 0 and 1, and

$$\oint_C \frac{4-3z}{z^2-z}dz = 2\pi i(-4) = -8\pi i$$

for any simple closed curve C for which $z=0$ lies inside C and $z=1$ lies outside, the integrations being taken in the counterclockwise sense.

Evaluation of real definite integrals

The residue theorem yields a simple and elegant method for evaluating certain classes of complicated real definite integrals. One serious restriction is that the contour must be closed. But many integrals of practical interest involve integration over open curves. Their paths of integration must be closed before the residue theorem can be applied. So our ability to evaluate such an integral depends crucially on how the contour is closed, since it requires knowledge of the additional contributions from the added parts of the closed contour. A number of techniques are known for closing open contours. The following types are most common in practice.

Improper integrals of the rational function $\int_{-\infty}^{\infty} f(x)dx$

The improper integral has the meaning

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx. \qquad (6.42)$$

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If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)dx. \tag{6.43}$$

We assume that the function $f(x)$ is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of the numerator. Then the limits in (6.42) exist and we can start from (6.43). We consider the corresponding contour integral

$$\oint_C f(z)dz,$$

along a contour C consisting of the line along the x-axis from $-r$ to r and the semicircle Γ above (or below) the x-axis having this line as its diameter (Fig. 6.16). Then let $r \to \infty$. If $f(x)$ is an even function this can be used to evaluate

$$\int_0^{\infty} f(x)dx.$$

Let us see why this works. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose r large enough, C encloses all these poles. Then by the residue theorem we have

$$\oint_C f(z)dz = \int_\Gamma f(z)dz + \int_{-r}^{r} f(x)dx = 2\pi i \sum \text{Res}\, f(z).$$

This gives

$$\int_{-r}^{r} f(x)dx = 2\pi i \sum \text{Res}\, f(z) - \int_\Gamma f(z)dz.$$

We next prove that $\int_\Gamma f(z)dz \to 0$ if $r \to \infty$. To this end, we set $z = re^{i\theta}$, then Γ is represented by $r = \text{const}$, and as z ranges along Γ, θ ranges from 0 to π. Since

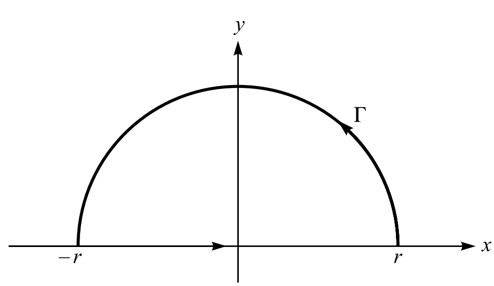

Figure 6.16. Path of the contour integral.

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the degree of the denominator of $f(z)$ is at least 2 units higher than the degree of the numerator, we have

$$|f(z)| < k/|z|^2 \qquad (|z| = r > r_0)$$

for sufficiently large constants k and r. By applying (6.24) we thus obtain

$$\left|\int_\Gamma f(z)dz\right| < \frac{k}{r^2}\pi r = \frac{k\pi}{r}.$$

Hence, as $r \to \infty$, the value of the integral over Γ approaches zero, and we obtain

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{Res}\, f(z). \tag{6.44}$$

Example 6.28
Using (6.44), show that

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Solution: $f(z) = 1/(1+z^4)$ has four simple poles at the points

$$z_1 = e^{\pi i/4}, \quad z_2 = e^{3\pi i/4}, \quad z_3 = e^{-3\pi i/4}, \quad z_4 = e^{-\pi i/4}.$$

The first two poles, z_1 and z_2, lie in the upper half-plane (Fig. 6.17) and we find, using L'Hospital's rule

$$\underset{z=z_1}{\text{Res}}\, f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4},$$

$$\underset{z=z_2}{\text{Res}}\, f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4},$$

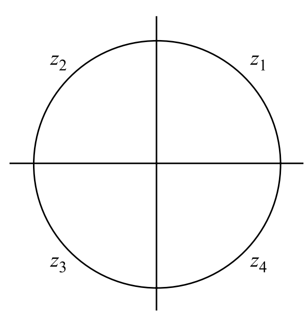

Figure 6.17.

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then

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{4}\left(-e^{\pi i/4} + e^{-\pi i/4}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

and so

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Example 6.29
Show that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$$

Solution: The poles of

$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$$

enclosed by the contour of Fig. 6.17 are $z = i$ of order 2 and $z = -1 + i$ of order 1.

The residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z^2}{(z+i)^{21}(z-i)^2(z^2+2z+2)}\right] = \frac{9i-12}{100}.$$

The residue at $z = -1 + i$ is

$$\lim_{z\to -1+i} (z+1-i)\ \frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = 2\pi i\left(\frac{9i-12}{100} + \frac{3-4i}{25}\right) = \frac{7\pi}{50}.$$

Integrals of the rational functions of $\sin\theta$ *and* $\cos\theta$ $\int_0^{2\pi} G(\sin\theta, \cos\theta) d\theta$

$G(\sin\theta, \cos\theta)$ is a real rational function of $\sin\theta$ and $\cos\theta$ finite on the interval $0 \le \theta \le 2\pi$. Let $z = e^{i\theta}$, then

$$dz = ie^{i\theta} d\theta, \quad \text{or} \quad d\theta = dz/iz, \quad \sin\theta = (z - z^{-1})/2i, \quad \cos\theta = (z + z^{-1})/2$$

and the given integrand becomes a rational function of z, say, $f(z)$. As θ ranges from 0 to 2π, the variable z ranges once around the unit circle $|z| = 1$ in the counterclockwise sense. The given integral takes the form

$$\oint_C f(z)\frac{dz}{iz},$$

the integration being taken in the counterclockwise sense around the unit circle.

Example 6.30
Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}.$$

Solution: Let $z = e^{i\theta}$, then $dz = ie^{i\theta}d\theta$, or $d\theta = dz/iz$, and

$$\sin\theta = \frac{z - z^{-1}}{2i}, \ \cos\ \theta = \frac{z + z^{-1}}{2},$$

then

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i},$$

where C is the circle of unit radius with its center at the origin (Fig. 6.18).

We need to find the poles of

$$\frac{1}{(1-2i)z^2 + 6iz - 1 - 2i}:$$

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$
$$= 2 - i, (2 - i)/5,$$

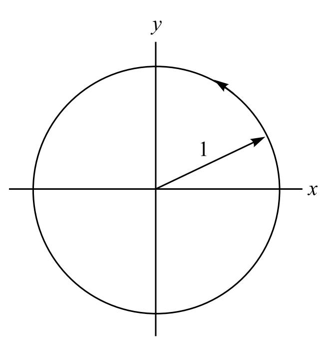

Figure 6.18.

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only $(2-i)/5$ lies inside C, and residue at this pole is

$$\lim_{z\to(2-i)/5}[z-(2-i)/5]\left[\frac{2}{(1-2i)z^2+6iz-1-2i}\right]$$
$$=\lim_{z\to(2-i)/5}\frac{2}{2(1-2i)z+6i}=\frac{1}{2i}\quad \text{by L'Hospital's rule.}$$

Then

$$\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}=\oint_C\frac{2dz}{(1-2i)z^2+6iz-1-2i}=2\pi i(1/2i)=\pi.$$

Fourier integrals of the form $\int_{-\infty}^{\infty}f(x)\begin{Bmatrix}\sin mx\\ \cos mx\end{Bmatrix}dx$

If $f(x)$ is a rational function satisfying the assumptions stated in connection with improper integrals of rational functions, then the above integrals may be evaluated in a similar way. Here we consider the corresponding integral

$$\oint_C f(z)e^{imz}dz$$

over the contour C as that in improper integrals of rational functions (Fig. 6.16), and obtain the formula

$$\int_{-\infty}^{\infty}f(x)e^{imx}dx=2\pi i\sum\text{Res}[f(z)e^{imz}]\quad (m>0), \tag{6.45}$$

where the sum consists of the residues of $f(z)e^{imz}$ at its poles in the upper half-plane. Equating the real and imaginary parts on each side of Eq. (6.45), we obtain

$$\int_{-\infty}^{\infty}f(x)\cos mx dx=-2\pi\sum\text{Im Res}[f(z)e^{imz}], \tag{6.46}$$

$$\int_{-\infty}^{\infty}f(x)\sin mx dx=2\pi\sum\text{Re Res}[f(z)e^{imz}]. \tag{6.47}$$

To establish Eq. (6.45) we should now prove that the value of the integral over the semicircle Γ in Fig. 6.16 approaches zero as $r\to\infty$. This can be done as follows. Since Γ lies in the upper half-plane $y\geq 0$ and $m>0$, it follows that

$$|e^{imz}|=|e^{imx}||e^{-my}|=e^{-my}\leq 1\qquad (y\geq 0, m>0).$$

From this we obtain

$$|f(z)e^{imz}|=|f(z)|\leq|e^{imz}||f(z)|\qquad (y\geq 0, m>0),$$

which reduces our present problem to that of an improper integral of a rational function of this section, since $f(x)$ is a rational function satisfying the assumptions stated in connection these improper integrals. Continuing as before, we see that the value of the integral under consideration approaches zero as r approaches ∞, and Eq. (6.45) is established.

Example 6.31
Show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{k^2 + x^2} dx = \frac{\pi}{k} e^{-km}, \qquad \int_{-\infty}^{\infty} \frac{\sin mx}{k^2 + x^2} dx = 0 \qquad (m > 0, k > 0).$$

Solution: The function $f(z) = e^{imz}/(k^2 + z^2)$ has a simple pole at $z = ik$ which lies in the upper half-plane. The residue of $f(z)$ at $z = ik$ is

$$\operatorname*{Res}_{z=ik} \frac{e^{imz}}{k^2 + z^2} = \left[\frac{e^{imz}}{2z}\right]_{z=\mathrm{i}k} = \frac{e^{-mk}}{2ik}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{k^2 + x^2}\, dx = 2\pi i\, \frac{e^{-mk}}{2ik} = \frac{\pi}{k} e^{-mk}$$

and this yields the above results.

Other types of real improper integrals

These are definite integrals

$$\int_A^B f(x) dx$$

whose integrand becomes infinite at a point a in the interval of integration, $\lim_{x \to a} |f(x)| = \infty$. This means that

$$\int_A^B f(x) dx = \lim_{\varepsilon \to 0} \int_A^{a-\varepsilon} f(x) dx + \lim_{\eta \to 0} \int_{a+\eta} f(x) dx,$$

where both ε and η approach zero independently and through positive values. It may happen that neither of these limits exists when $\varepsilon, \eta \to 0$ independently, but

$$\lim_{\varepsilon \to 0} \left[\int_A^{a-\varepsilon} f(x) dx + \int_{a+\varepsilon}^B f(x) dx \right]$$

exists; this is called Cauchy's principal value of the integral and is often written

$$\text{pr. v.} \int_A^B f(x) dx.$$

To evaluate improper integrals whose integrands have poles on the real axis, we can use a path which avoids these singularities by following small semicircles with centers at the singular points. We now illustrate the procedure with a simple example.

Example 6.32
Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Solution: The function $\sin(z)/z$ does not behave suitably at infinity. So we consider e^{iz}/z, which has a simple pole at $z = 0$, and integrate around the contour C or $ABDEFGA$ (Fig. 6.19). Since e^{iz}/z is analytic inside and on C, it follows from Cauchy's integral theorem that

$$\oint_C \frac{e^{iz}}{z}\,dz = 0$$

or

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{C_2} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{C_1} \frac{e^{iz}}{z}\,dz = 0. \tag{6.48}$$

We now prove that the value of the integral over large semicircle C_1 approaches zero as R approaches infinity. Setting $z = Re^{i\theta}$, we have $dz = iRe^{i\theta}d\theta, dz/z = id\theta$ and therefore

$$\left|\int_{C_1} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^\pi e^{iz}id\theta\right| \le \int_0^\pi \left|e^{iz}\right|d\theta.$$

In the integrand on the right,

$$\left|e^{iz}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right| = e^{-R\sin\theta}.$$

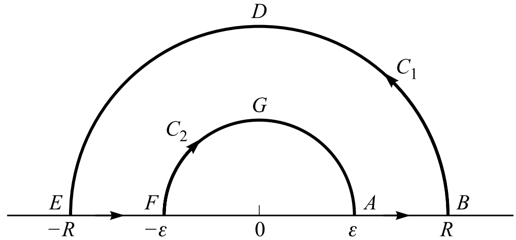

Figure 6.19.

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By inserting this and using $\sin(\pi - \theta) = \sin\theta$ we obtain

$$\int_0^{\pi} |e^{iz}| d\theta = \int_0^{\pi} e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta$$
$$= 2\left[\int_0^{\varepsilon} e^{-R\sin\theta} d\theta + \int_{\varepsilon}^{\pi/2} e^{-R\sin\theta} d\theta\right],$$

where ε has any value between 0 and $\pi/2$. The absolute values of the integrands in the first and the last integrals on the right are at most equal to 1 and $e^{-R\sin\varepsilon}$, respectively, because the integrands are monotone decreasing functions of θ in the interval of integration. Consequently, the whole expression on the right is smaller than

$$2\left[\int_0^{\varepsilon} d\theta + e^{-R\sin\theta}\int_{\varepsilon}^{\pi/2} d\theta\right] = 2\left[\varepsilon + e^{-R\sin\theta}\left(\frac{\pi}{2} - \varepsilon\right)\right] < 2\varepsilon + \pi e^{-R\sin\varepsilon}.$$

Altogether

$$\left|\int_{C_1} \frac{e^{iz}}{z} dz\right| < 2\varepsilon + \pi e^{-R\sin\varepsilon}.$$

We first take ε arbitrarily small. Then, having fixed ε, the last term can be made as small as we please by choosing R sufficiently large. Hence the value of the integral along C_1 approaches 0 as $R \to \infty$.

We next prove that the value of the integral over the small semicircle C_2 approaches zero as $\varepsilon \to 0$. Let $z = \varepsilon^{i\theta}$, then

$$\int_{C_2} \frac{e^{iz}}{z} dz = -\lim_{\varepsilon\to 0}\int_{\pi}^{0} \frac{\exp(i\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta = -\lim_{\varepsilon\to 0}\int_{\pi}^{0} i\exp(i\varepsilon e^{i\theta}) d\theta = \pi i$$

and Eq. (6.48) reduces to

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \pi i + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx = 0.$$

Replacing x by $-x$ in the first integral and combining with the last integral, we find

$$\int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x} dx + \pi i = 0.$$

Thus we have

$$2i\int_{\varepsilon}^{R} \frac{\sin x}{x} dx = \pi i.$$

Taking the limits $R \to \infty$ and $\varepsilon \to 0$

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Problems

6.1. Given three complex numbers $z_1 = a + ib$, $z_2 = c + id$, and $z_3 = g + ih$, show that:

(a) $z_1 + z_2 = z_2 + z_1$ commutative law of addition;

(b) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ associative law of addition;

(c) $z_1 z_2 = z_2 z_1$ commutative law of multiplication;

(d) $z_1(z_2 z_3) = (z_1 z_2) z_3$ associative law of multiplication.

6.2. Given

$$z_1 = \frac{3+4i}{3-4i}, \qquad z_2 = \left[\frac{1+2i}{1-3i}\right]^2$$

find their polar forms, complex conjugates, moduli, product, the quotient z_1/z_2.

6.3. The absolute value or modulus of a complex number $z = x + iy$ is defined as

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}.$$

If $z_1, z_2, \ldots, z_m$ are complex numbers, show that the following hold:

(a) $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$.

(b) $|z_1/z_2| = |z_1|/|z_2|$ if $z_2 \neq 0$.

(c) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(d) $|z_1 + z_2| \geq |z_1| - |z_2|$ or $|z_1 - z_2| \geq |z_1| - |z_2|$.

6.4 Find all roots of (a) $\sqrt[5]{-32}$, and (b) $\sqrt[3]{1+i}$, and locate them in the complex plane.

6.5 Show, using De Moivre's theorem, that:

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;

(b) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

6.6 Given $z = re^{i\theta}$, interpret $ze^{i\theta}$, where α is real geometrically.

6.7 Solve the quadratic equation $az^2 + bz + c = 0, a \neq 0$.

6.8 A point P moves in a counterclockwise direction around a circle of radius 1 with center at the origin in the z plane. If the mapping function is $w = z^2$, show that when P makes one complete revolution the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle of radius 1 with center at the origin.

6.9 Show that $f(z) = \ln z$ has a branch point at $z = 0$.

6.10 Let $w = f(z) = (z^2 + 1)^{1/2}$, show that:

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(*a*) $f(z)$ has branch points at $z = \pm I$.

(*b*) a complete circuit around both branch points produces no change in the branches of $f(z)$.

6.11 Apply the definition of limits to prove that:

$$\lim_{z\to 1}\frac{z^2-1}{z-1} = 2.$$

6.12. Prove that:

(*a*) $f(z) = z^2$ is continuous at $z = z_0$, and

(*b*) $f(z) = \begin{cases} z^2, z \neq z_0 \\ 0, z = z_0 \end{cases}$ is discontinuous at $z = z_0$, where $z_0 \neq 0$.

6.13 Given $f(z) = z^*$, show that $f'(i)$ does not exist.

6.14 Using the definition, find the derivative of $f(z) = z^3 - 2z$ at the point where: (*a*) $z = z_0$, and (b) $z = -1$.

6.15. Show that f is an analytic function of z if it does not depend on $z^*: f(z, z^*) = f(z)$. In other words, $f(x, y) = f(x + iy)$, that is, x and y enter f only in the combination $x + iy$.

6.16. (a) Show that $u = y^3 - 3x^2y$ is harmonic.

(*b*) Find v such that $f(z) = u + iv$ is analytic.

6.17 (*a*) If $f(z) = u(x, y) + iv(x, y)$ is analytic in some region R of the z plane, show that the one-parameter families of curves $u(x, y) = C_1$ and $v(x, y) = C_2$ are orthogonal families.

(*b*) Illustrate (*a*) by using $f(z) = z^2$.

6.18 For each of the following functions locate and name the singularities in the finite z plane:

(*a*) $f(z) = \dfrac{z}{(z^2+4)^4}$; (*b*) $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$; (*c*) $f(z) = \sum_{n=0}^{\infty}\dfrac{1}{z^n n!}$.

6.19 (*a*) Locate and name all the singularities of

$$f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}.$$

(*b*) Determine where $f(z)$ is analytic.

6.20 (*a*) Given $e^z = e^x(\cos y + i\sin y)$, show that $(d/dz)e^z = e^z$.

(*b*) Show that $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

(Hint: set $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ and apply the addition formulas for the sine and cosine.)

6.21 Show that: (*a*) $\ln e^z = z + 2n\pi i$, (*b*) $\ln z_1/z_2 = \ln z_1 - \ln z_2 + 2n\pi i$.

6.22 Find the values of: (a) $\ln i$, (*b*) $\ln(1 - i)$.

6.23 Evaluate $\int_C z^* dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by:

(*a*) $z = t^2 + it$;

(*b*) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

6.24 Evaluate $\oint_C dz/(z-a)^n$, $n = 2, 3, 4, \ldots$ where $z = a$ is inside the simple closed curve C.

6.25 If $f(z)$ is analytic in a simply-connected region R, and a and z are any two points in R, show that the integral

$$\int_a^z f(z)dz$$

is independent of the path in R joining a and z.

6.26 Let $f(z)$ be continuous in a simply-connected region R and let a and z be points in R. Prove that $F(z) = \int_a^z f(z')dz'$ is analytic in R, and $F'(z) = f(z)$.

6.27 Evaluate

(a) $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

(b) $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4} dz,$

where C is the circle $|z| = 1$.

6.28 Evaluate

$$\oint_C \frac{2 \sin z^2}{(z-1)^4} dz,$$

where C is any simple closed path not passing through 1.

6.29 Show that the complex sequence

$$z_n = \frac{1}{n} - \frac{n^2 - 1}{n} i$$

diverges.

6.30 Find the region of convergence of the series $\sum_{n=1}^{\infty} (z+2)^{n+1}/(n+1)^3 4^n$.

6.31 Find the Maclaurin series of $f(z) = 1/(1+z^2)$.

6.32 Find the Taylor series of $f(z) = \sin z$ about $z = \pi/4$, and determine its circle of convergence. (Hint: $\sin z = \sin[a + (z-a)]$.)

6.33 Find the Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $\displaystyle (z-3) \sin \frac{1}{z+2}, \quad z = -2;$

(b) $\displaystyle \frac{z}{(z+1)(z+2)}, \quad z = -2;$

(c) $\displaystyle \frac{1}{z(z-3)^2}, \quad z = 3.$

6.34 Expand $f(z) = 1/[(z+1)(z+3)]$ in a Laurent series valid for:
(a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$.

6.35 Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}, \qquad a > 0, b > 0.$$

6.36 Evaluate

(*a*)
$$\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2},$$

where p is a fixed number in the interval $0 < p < 1$;

(*b*)
$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}.$$

6.37 Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx.$$

6.38 Show that:

(*a*)
$$\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}};$$

(*b*)
$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}, \qquad 0 < p < 1.$$

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