7

Special functions of mathematical physics

The functions discussed in this chapter arise as solutions of second-order differential equations which appear in special, rather than in general, physical problems. So these functions are usually known as the special functions of mathematical physics. We start with Legendre's equation (Adrien Marie Legendre, 1752–1833, French mathematician).

Legendre's equation

Legendre's differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu + 1)y = 0, \tag{7.1}$$

where ν is a positive constant, is of great importance in classical and quantum physics. The reader will see this equation in the study of central force motion in quantum mechanics. In general, Legendre's equation appears in problems in classical mechanics, electromagnetic theory, heat, and quantum mechanics, with spherical symmetry.

Dividing Eq. (7.1) by $1 - x^2$, we obtain the standard form

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{\nu(\nu+1)}{1-x^2}y = 0.$$

We see that the coefficients of the resulting equation are analytic at $x = 0$, so the origin is an ordinary point and we may write the series solution in the form

$$y = \sum_{m=0}^{\infty} a_m x^m. \tag{7.2}$$

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Substituting this and its derivatives into Eq. (7.1) and denoting the constant $\nu(\nu+1)$ by k we obtain

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first term as two separate series we have

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0,$$

which can be written as:

$$\begin{aligned}
&2\times 1a_2 + 3\times 2a_3x + 4\times 3a_4x^2 + \cdots + (s+2)(s+1)a_{s+2}x^s + \cdots \\
&\qquad\qquad -2\times 1a_2x^2 - \cdots \qquad - (s(s-1)a_sx^s - \cdots \\
&\qquad -2\times 1a_1x - 2\times 2a_2x^2 - \cdots \qquad - 2sa_sx^s - \cdots \\
&+ka_0 \quad + ka_1x \quad + ka_2x^2 + \cdots \qquad + ka_sx^s + \cdots = 0.
\end{aligned}$$

Since this must be an identity in x if Eq. (7.2) is to be a solution of Eq. (7.1), the sum of the coefficients of each power of x must be zero; remembering that $k = \nu(\nu+1)$ we thus have

$$2a_2 + \nu(\nu+1)a_0 = 0, \tag{7.3a}$$

$$6a3 + [-2 + v(v+1)]a_1 = 0, \tag{7.3b}$$

and in general, when $s = 2, 3, \ldots,$

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + \nu(\nu{+}1)]a_s = 0. \tag{4.4}$$

The expression in square brackets [...] can be written

$$(\nu - s)(\nu + s + 1).$$

We thus obtain from Eq. (7.4)

$$a_{s+2} = -\frac{(\nu-s)(\nu+s+1)}{(s+2)(s+1)}a_s \qquad (s = 0, 1, \ldots). \tag{7.5}$$

This is a recursion formula, giving each coefficient in terms of the one two places before it in the series, except for a_0 and a_1, which are left as arbitrary constants.

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We find successively

$$a_2 = -\frac{\nu(\nu+1)}{2!}a_0, \qquad a_3 = -\frac{(\nu-1)(\nu+2)}{3!}a_1,$$

$$a_4 = -\frac{(\nu-2)(\nu+3)}{4\cdot 3}a_2, \qquad a_5 = -\frac{(\nu-3)(\nu+4)}{3!}a_3,$$

$$= \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!}a_0, \qquad = \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!}a_1,$$

etc. By inserting these values for the coefficients into Eq. (7.2) we obtain

$$y(x) = a_0 y_1(x) + a_1 y_2(x), \tag{7.6}$$

where

$$y_1(x) = 1 - \frac{\nu(\nu+1)}{2!}x^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!}x^4 - + \cdots \tag{7.7a}$$

and

$$y_2(x) = x = \frac{(\nu-1)(\nu+2)}{3!}x^3 + \frac{(\nu-2)(\nu-1)(\nu+2)(\nu+4)}{5!}x^5 - + \cdots. \tag{7.7b}$$

These series converge for $|x| < 1$. Since Eq. (7.7a) contains even powers of x, and Eq. (7.7b) contains odd powers of x, the ratio y_1/y_2 is not a constant, and y_1 and y_2 are linearly independent solutions. Hence Eq. (7.6) is a general solution of Eq. (7.1) on the interval $-1 < x < 1$.

In many applications the parameter ν in Legendre's equation is a positive integer n. Then the right hand side of Eq. (7.5) is zero when $s = n$ and, therefore, $a_{n+2} = 0$ and $a_{n+4} = 0, \ldots$. Hence, if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, the same is true with respect to $y_2(x)$. These polynomials, multiplied by some constants, are called Legendre polynomials. Since they are of great practical importance, we will consider them in some detail. For this purpose we rewrite Eq. (7.5) in the form

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)}a_{s+2} \tag{7.8}$$

and then express all the non-vanishing coefficients in terms of the coefficient a_n of the highest power of x of the polynomial. The coefficient a_n is then arbitrary. It is customary to choose $a_n = 1$ when $n = 0$ and

$$a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1\times 3\times 5\cdots(2n-1)}{n!}, \qquad n = 1, 2, \ldots, \tag{7.9}$$

the reason being that for this choice of a_n all those polynomials will have the value 1 when $x = 1$. We then obtain from Eqs. (7.8) and (7.9)

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n = -\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}$$
$$= -\frac{n(n-1)2n(2n-1)(2n-2)!!}{2(2n-1)2^n n(n-1)!n(n-1)(n-2)!},$$

that is,

$$a_{n-2} = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}.$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

etc., and in general

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}. \tag{7.10}$$

The resulting solution of Legendre's equation is called the Legendre polynomial of degree n and is denoted by $P_n(x)$; from Eq. (7.10) we obtain

$$\begin{aligned} P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n(n!)^2}x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!}x^{n-2} + - \cdots, \end{aligned} \tag{7.11}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. In particular (Fig. 7.1)

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x).$$

Rodrigues' formula for $P_n(x)$

The Legendre polynomials $P_n(x)$ are given by the formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]. \tag{7.12}$$

We shall establish this result by actually carrying out the indicated differentiations, using the Leibnitz rule for nth derivative of a product, which we state below without proof:

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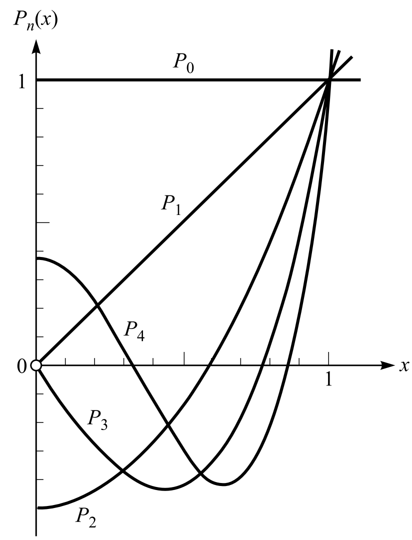

Figure 7.1. Legendre polynomials.

If we write $D^n u$ as u_n and $D^n v$ as v_n, then

$$(uv)_n = uv_n + {}^nC_1 u_1 v_{n-1} + \cdots + {}^nC_r u_r v_{n-r} + \cdots + u_n v,$$

where $D = d/dx$ and ${}^nC_r$ is the binomial coefficient and is equal to $n!/[r!(n-r)!]$.

We first notice that Eq. (7.12) holds for $n = 0$, 1. Then, write

$$z = (x^2 - 1)^n / 2^n n!$$

so that

$$(x^2 - 1)Dz = 2nxz. \tag{7.13}$$

Differentiating Eq. (7.13) $(n+1)$ times by the Leibnitz rule, we get

$$(1 - x^2)D^{n+2}z - 2xD^{n+1}z + n(n+1)D^n z = 0.$$

Writing $y = D^n z$, we then have:

(i) y is a polynomial.
(ii) The coefficient of x^n in $(x^2 - 1)^n$ is $(-1)^{n/2}\, {}^nC_{n/2}$ (n even) or 0 (n odd). Therefore the lowest power of x in $y(x)$ is x^0 (n even) or x^1 (n odd). It follows that

$$y_n(0) = 0 \qquad (n \text{ odd})$$

and

$$y_n(0) = \frac{1}{2^n n!}(-1)^{n/2}\, {}^nC_{n/2} n! = \frac{(-1)^{n/2} n!}{2^n[(n/2)!]^2} \quad (n \text{ even}).$$

By Eq. (7.11) it follows that

$$y_n(0) = P_n(0) \quad (\text{all } n).$$

(*iii*) $(1 - x^2)D^2y - 2xDy + n(n+1)y = 0$, which is Legendre's equation.

Hence Eq. (7.12) is true for all n.

The generating function for $P_n(x)$

One can prove that the polynomials $P_n(x)$ are the coefficients of z^n in the expansion of the function $\Phi(x, z) = (1 - 2xz + z^2)^{-1/2}$, with $|z| < 1$; that is,

$$\Phi(x, z) = (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n, \qquad |z| < 1. \tag{7.14}$$

$\Phi(x, z)$ is called the generating function for Legendre polynomials $P_n(x)$. We shall be concerned only with the case in which

$$x = \cos\theta \quad (-\pi < \theta \le \pi)$$

and then

$$z^2 - 2xz + 1 \equiv (z - e^{i\theta})(z - e^{i\theta}).$$

The expansion (7.14) is therefore possible when $|z| < 1$. To prove expansion (7.14) we have

$$\text{lhs} = 1 + \tfrac{1}{2}z(2x - 1) + \frac{1 \times 3}{2^2 \times 2!}z^2(2x - z)^2 + \cdots + \frac{1 \times 3 \cdots (2n-1)}{2^n n!}z^n(2x - z)^n + \cdots.$$

The coefficient of z^n in this power series is

$$\frac{1 \times 3 \cdots (2n-1)}{2^n n!}(2^n x^n) + \frac{1 \times 3 \cdots (2n-3)}{2^{n-1}(n-1)!}[-(n-1)(2x)^{n-2}] + \cdots = P_n(x)$$

by Eq. (7.11). We can use Eq. (7.14) to find successive polynomials explicitly. Thus, differentiating Eq. (7.14) with respect to z so that

$$(x - z)(1 - 2xz + z^2)^{-3/2} = \sum_{n=1}^{\infty} nz^{n-1}P_n(x)$$

and using Eq. (7.14) again gives

$$(x - z)\left[P_0(x) + \sum_{n=1}^{\infty} P_n(x)z^n\right] = (1 - 2xz + z^2)\sum_{n=1}^{\infty} nz^{n-1}P_n(x). \tag{7.15}$$

Then expanding coefficients of z^n in Eq. (7.15) leads to the recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x). \qquad (7.16)$$

This gives P_4, P_5, P_6, etc. very quickly in terms of P_0, P_1, and P_3.

Recurrence relations are very useful in simplifying work, helping in proofs or derivations. We list four more recurrence relations below without proofs or derivations:

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x); \qquad (7.16a)$$

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x); \qquad (7.16b)$$

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x); \qquad (7.16c)$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x). \qquad (7.16d)$$

With the help of the recurrence formulas (7.16) and (7.16b), it is straightforward to establish the other three. Omitting the full details, which are left for the reader, these relations can be obtained as follows:

(*i*) differentiation of Eq. (7.16) with respect to x and the use of Eq. (7.16b) to eliminate $P_{n+1}'(x)$ leads to relation (7.16a);

(*ii*) the addition of Eqs. (7.16a) and (7.16b) immediately yields relation (7.16d);

(*iii*) the elimination of $P_{n-1}'(x)$ between Eqs. (7.16b) and (7.16a) gives relation (7.16c).

Example 7.1

The physical significance of expansion (7.14) is apparent in this simple example: find the potential V of a point charge at point P due to a charge $+q$ at Q.

Solution: Suppose the origin is at O (Fig. 7.2). Then

$$V_P = \frac{q}{R} = q(\rho^2 - 2r\rho\cos\theta + r^2)^{-1/2}.$$

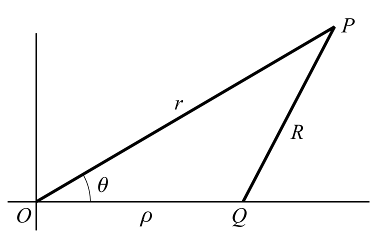

Figure 7.2.

Thus, if $r < \rho$

$$V_P = \frac{q}{\rho}(1 - 2z\cos\theta + z^2)^{-1/2}, \qquad z = r/\rho,$$

which gives

$$V_P = \frac{q}{\rho}\sum_{n=0}^{\infty}\left(\frac{r}{\rho}\right)^n P_n(\cos\theta) \qquad (r < \rho).$$

Similarly, when $r > \rho$, we get

$$V_P = \frac{q}{\rho}\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^{n+1} P_n(\cos\theta).$$

There are many problems in which it is essential that the Legendre polynomials be expressed in terms of θ, the colatitude angle of the spherical coordinate system. This can be done by replacing x by $\cos\theta$. But this will lead to expressions that are quite inconvenient because of the powers of $\cos\theta$ they contain. Fortunately, using the generating function provided by Eq. (7.14), we can derive more useful forms in which cosines of multiples of θ take the place of powers of $\cos\theta$. To do this, let us substitute

$$x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$$

into the generating function, which gives

$$[1 - z(e^{i\theta} + e^{-i\theta}) + z^2]^{-1/2} = [(1 - ze^{i\theta})(1 - ze^{-i\theta})]^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos\theta)z^n.$$

Now by the binomial theorem, we have

$$(1 - ze^{i\theta})^{-1/2} = \sum_{n=0}^{\infty} a_n z^n e^{ni\theta}, \qquad (1 - ze^{-i\theta})^{-1/2} = \sum_{n=0}^{\infty} a_n z^n e^{-ni\theta},$$

where

$$a_n = \frac{1 \times 3 \times 5 \cdots (2n-1)}{2 \times 4 \times 6 \cdots (2n)}, \qquad n \geq 1, \qquad a_0 = 1. \tag{7.17}$$

To find the coefficient of z^n in the product of these two series, we need to form the Cauchy product of these two series. What is a Cauchy product of two series? We state it below for the reader who is in need of a review:

> The Cauchy product of two infinite series, $\sum_{n=0}^{\infty} u_n(x)$ and $\sum_{n=0}^{\infty} v_n(x)$, is defined as the sum over n
>
> $$\sum_{n=0}^{\infty} s_n(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} u_k(x)v_{n-k}(x),$$

where $s_n(x)$ is given by

$$s_n(x) = \sum_{k=0}^{n} u_k(x)v_{n-k}(x) = u_0(x)v_n(x) + \cdots + u_n(x)v_0(x).$$

Now the Cauchy product for our two series is given by

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(a_{n-k}z^{n-k}e^{(n-k)i\theta}\right)\left(a_k z^k e^{-ki\theta}\right) = \sum_{n=0}^{\infty}\left(z^n \sum_{k=0}^{n} a_k a_{n-k} e^{(n-2k)i\theta}\right). \tag{7.18}$$

In the inner sum, which is the sum of interest to us, it is straightforward to prove that, for $n \geq 1$, the terms corresponding to $k = j$ and $k = n - j$ are identical except that the exponents on e are of opposite sign. Hence these terms can be paired, and we have for the coefficient of z^n,

$$\begin{aligned} P_n(\cos\theta) &= a_0 a_n (e^{ni\theta} + e^{-ni\theta}) + a_1 a_{n-1}(e^{(n-2)i\theta} + e^{-(n-2)i\theta}) + \cdots \\ &= 2[a_0 a_n \cos n\theta + a_1 a_{n-1}\cos(n-2)\theta + \cdots]. \end{aligned} \tag{7.19}$$

If n is odd, the number of terms is even and each has a place in one of the pairs. In this case, the last term in the sum is

$$a_{(n-1)/2}a_{(n+1)/2}\cos\theta.$$

If n is even, the number of terms is odd and the middle term is unpaired. In this case, the series (7.19) for $P_n(\cos\theta)$ ends with the constant term

$$a_{n/2}a_{n/2}.$$

Using Eq. (7.17) to compute values of the a_n, we find from the unit coefficient of z^0 in Eqs. (7.18) and (7.19), whether n is odd or even, the specific expressions

$$\left.\begin{aligned} &P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta, \quad P_2(\cos\theta) = (3\cos 2\theta + 1)/4 \\ &P_3(\cos\theta) = (5\cos 3\theta + 3\cos\theta)/8 \\ &P_4(\cos\theta) = (35\cos 4\theta + 20\cos 2\theta + 9)/64 \\ &P_5(\cos\theta) = (63\cos 5\theta + 35\cos 3\theta + 30\cos\theta)/128 \\ &P_6(\cos\theta) = (231\cos 6\theta + 126\cos 4\theta + 105\cos 2\theta + 50)/512 \end{aligned}\right\}. \tag{7.20}$$

Orthogonality of Legendre polynomials

The set of Legendre polynomials $\{P_n(x)\}$ is orthogonal for $-1 \leq x \leq +1$. In particular we can show that

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \begin{cases} 2/(2n+1) & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \tag{7.21}$$

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(*i*) $m \neq n$: Let us rewrite the Legendre equation (7.1) for $P_m(x)$ in the form

$$\frac{d}{dx}\left[(1-x^2)P_m'(x)\right] + m(m+1)P_m(x) = 0 \tag{7.22}$$

and the one for $P_n(x)$

$$\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0. \tag{7.23}$$

We then multiply Eq. (7.22) by $P_n(x)$ and Eq. (7.23) by $P_m(x)$, and subtract to get

$$P_m \frac{d}{dx}\left[(1-x^2)P_n'\right] - P_n \frac{d}{dx}\left[(1-x^2)P_m'\right] + [n(n+1) - m(m+1)]P_m P_n = 0.$$

The first two terms in the last equation can be written as

$$\frac{d}{dx}\left[(1-x^2)(P_m P_n' - P_n P_m')\right].$$

Combining this with the last equation we have

$$\frac{d}{dx}\left[(1-x^2)(P_m P_n' - P_n P_m')\right] + [n(n+1) - m(m+1)]P_m P_n = 0.$$

Integrating the above equation between -1 and 1 we obtain

$$(1-x^2)(P_m P_n' - P_n P_m')|_{-1}^{1} + [n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

The integrated term is zero because $(1-x^2) = 0$ at $x = \pm 1$, and $P_m(x)$ and $P_n(x)$ are finite. The bracket in front of the integral is not zero since $m \neq n$. Therefore the integral must be zero and we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \qquad m \neq n.$$

(*ii*) $m = n$: We now use the recurrence relation (7.16a), namely

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Multiplying this recurrence relation by $P_n(x)$ and integrating between -1 and 1, we obtain

$$n\int_{-1}^{1} [P_n(x)]^2 dx = \int_{-1}^{1} xP_n(x)P_n'(x)dx - \int_{-1}^{1} P_n(x)P_{n-1}'(x)dx. \tag{7.24}$$

The second integral on the right hand side is zero. (Why?) To evaluate the first integral on the right hand side, we integrate by parts

$$\int_{-1}^{1} xP_n(x)P_n'(x)dx = \frac{x}{2}[P_n(x)]^2|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}[P_n(x)]^2 dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_n(x)]^2 dx.$$

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Substituting these into Eq. (7.24) we obtain

$$n\int_{-1}^{1}[P_n(x)]^2dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_n(x)]^2dx,$$

which can be simplified to

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2}{2n+1}.$$

Alternatively, we can use generating function

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)z^n.$$

We have on squaring both sides of this:

$$\frac{1}{1-2xz+z^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)z^{m+n}.$$

Then by integrating from -1 to 1 we have

$$\int_{-1}^{1}\frac{dx}{1-2xz+z^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_m(x)P_n(x)dx\right\}z^{m+n}.$$

Now

$$\int_{-1}^{1}\frac{dx}{1-2xz+z^2} = -\frac{1}{2z}\int_{-1}^{1}\frac{d(1-2xz+z^2)}{1-2xz+z^2} = -\frac{1}{2z}\ln(1-2xz+z^2)|_{-1}^{1}$$

and

$$\int_{-1}^{1}P_m(x)P_n(x)dx = 0, \qquad m \neq n.$$

Thus, we have

$$-\frac{1}{2z}\ln(1-2xz+z^2)|_{-1}^{1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)dx\right\}z^{2n}$$

or

$$\frac{1}{z}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)dx\right\}z^{2n},$$

that is,

$$\sum_{n=0}^{\infty}\frac{2z^{2n}}{2n+1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)dx\right\}z^{2n}.$$

Equating coefficients of z^{2n} we have as required $\int_{-1}^{1}P_n^2(x)dx = 2/(2n+1)$.

Since the Legendre polynomials form a complete orthogonal set on (−1, 1), we can expand functions in Legendre series just as we expanded functions in Fourier series:

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x).$$

The coefficients c_i can be found by a method parallel to the one we used in finding the formulas for the coefficients in a Fourier series. We shall not pursue this line further.

There is a second solution of Legendre's equation. However, this solution is usually only required in practical applications in which $|x| > 1$ and we shall only briefly discuss it for such values of x. Now solutions of Legendre's equation relative to the regular singular point at infinity can be investigated by writing $x^2 = t$. With this substitution,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 2t^{1/2}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} + 4t\frac{d^2y}{dt^2},$$

and Legendre's equation becomes, after some simplifications,

$$t(1-t)\frac{d^2y}{dt^2} + \left(\frac{1}{2} - \frac{3}{2}t\right)\frac{dy}{dt} + \frac{\nu(\nu+1)}{4}y = 0.$$

This is the hypergeometric equation with $\alpha = -\nu/2, \beta = (1+\nu)/2$, and $\gamma = \frac{1}{2}$:

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta y = 0;$$

we shall not seek its solutions. The second solution of Legendre's equation is commonly denoted by $Q_\nu(x)$ and is called the Legendre function of the second kind of order ν. Thus the general solution of Legendre's equation (7.1) can be written

$$y = AP_\nu(x) + BQ_\nu(x),$$

A and B being arbitrary constants. $P_\nu(x)$ is called the Legendre function of the first kind of order ν and it reduces to the Legendre polynomial $P_n(x)$ when ν is an integer n.

The associated Legendre functions

These are the functions of integral order which are solutions of the associated Legendre equation

$$(1-x^2)y'' - 2xy' + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0 \qquad (7.25)$$

with $m^2 \leq n^2$.

We could solve Eq. (7.25) by series; but it is more useful to know how the solutions are related to Legendre polynomials, so we shall proceed in the following way. We write

$$y = (1 - x^2)^{m/2} u(x)$$

and substitute into Eq. (7.25) whence we get, after a little simplification,

$$(1 - x^2)u'' - 2(m+1)xu' + [n(n+1) - m(m+1)]u = 0. \tag{7.26}$$

For $m = 0$, this is a Legendre equation with solution $P_n(x)$. Now we differentiate Eq. (7.26) and get

$$(1 - x^2)(u')'' - 2[(m+1)+1]x(u')' + [n(n+1) - (m+1)(m+2)]u' = 0. \tag{7.27}$$

Note that Eq. (7.27) is just Eq. (7.26) with u' in place of u, and $(m+1)$ in place of m. Thus, if $P_n(x)$ is a solution of Eq. (7.26) with $m = 0$, $P_n'(x)$ is a solution of Eq. (7.26) with $m = 1$, $P_n''(x)$ is a solution with $m = 2$, and in general for integral $m, 0 \le m \le n, (d^m/dx^m)P_n(x)$ is a solution of Eq. (7.26). Then

$$y = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \tag{7.28}$$

is a solution of the associated Legendre equation (7.25). The functions in Eq. (7.28) are called associated Legendre functions and are denoted by

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x). \tag{7.29}$$

Some authors include a factor $(-1)^m$ in the definition of $P_n^m(x)$.

A negative value of m in Eq. (7.25) does not change m^2, so a solution of Eq. (7.25) for positive m is also a solution for the corresponding negative m. Thus many references define $P_n^m(x)$ for $-n \le m \le n$ as equal to $P_n^{|m|}(x)$.

When we write $x = \cos\theta$, Eq. (7.25) becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}y = 0 \tag{7.30}$$

and Eq. (7.29) becomes

$$P_n^m(\cos\theta) = \sin^m\theta \frac{d^m}{d(\cos\theta)^m}\{P_n(\cos\theta)\}.$$

In particular

$$D^{-1} \text{ means } \int_1^x P_n(x)dx.$$

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Orthogonality of associated Legendre functions

As in the case of Legendre polynomials, the associated Legendre functions $P_n^m(x)$ are orthogonal for $-1 \le x \le 1$ and in particular

$$\int_{-1}^{1} P_m^s(x) P_n^s(x) dx = \frac{(n+s)!}{(n-s)!} \delta_{mn}. \tag{7.31}$$

To prove this, let us write for simplicity

$$M = P_s^m(x), \quad \text{and} \quad N = P_n^s(x)$$

and from Eq. (7.25), the associated Legendre equation, we have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dM}{dx}\right\} + \left\{m(m+1) - \frac{s^2}{1-x^2}\right\} M = 0 \tag{7.32}$$

and

$$\frac{d}{dx}\left\{(1-x^2)\frac{dN}{dx}\right\} + \left\{n(n+1) - \frac{s^2}{1-x^2}\right\} N = 0. \tag{7.33}$$

Multiplying Eq. (7.32) by N, Eq. (7.33) by M and subtracting, we get

$$M\frac{d}{dx}\left\{(1-x^2)\frac{dN}{dx}\right\} - N\frac{d}{dx}\left\{(1-x^2)\frac{dM}{dx}\right\} = \{m(m+1) - n(n+1)\}MN.$$

Integration between -1 and 1 gives

$$(m-n)(m+n-1)\int_{-1}^{1} MN dx = \int_{-1}^{1}\left[M\frac{d}{dx}\left\{(1-x^2)\frac{dN}{dx}\right\} - N\frac{d}{dx}\left\{(1-x^2)\frac{dM}{dx}\right\}\right] dx. \tag{7.34}$$

Integration by parts gives

$$\int_{-1}^{1} M\frac{d}{dx}\{(1-x^2)N'\}dx = [MN'(1-x^2)]_{-1}^{1} - \int_{-1}^{1}(1-x^2)M'N'dx$$
$$= -\int_{-1}^{1}(1-x^2)M'N'dx.$$

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Then integrating by parts once more, we obtain

$$\int_{-1}^{1} M\frac{d}{dx}\{(1-x^2)N'\}dx = -\int_{-1}^{1}(1-x^2)M'N'dx$$

$$= -[MN(1-x^2)]_{-1}^{1} + \int_{-1}^{1} N\frac{d}{dx}\{(1-x^2)M'\}dx$$

$$= \int_{-1}^{1} N\frac{d}{dx}\{(1-x^2)M'\}dx.$$

Substituting this in Eq. (7.34) we get

$$(m-n)(m+n-1)\int_{-1}^{1} MNdx = 0.$$

If $m \le n$, we have

$$\int_{-1}^{1} MNdx = \int_{-1}^{1} P_m^s(x)P_m^s(x)dx = 0 \qquad (m \ne n).$$

If $m = n$, let us write

$$P_n^s(x) = (1-x^2)^{s/2}\frac{d^s}{dx^s}P_n(x) = \frac{(1-x^2)^{s/2}}{2^n n!}\frac{d^{s+n}}{dx^{s+n}}\{(x^2-1)^n\}.$$

Hence

$$\int_{-1}^{1} P_n^s(x)P_n^s(x)dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^s D^{n+s}\{(x^2-1)^n\}D^{n+s}$$
$$\times\{(x^2-1)^n\}dx, \qquad (D^k = d^k/dx^k).$$

Integration by parts gives

$$\frac{1}{2^{2n}(n!)^2}[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}D^{n+s-1}\{(x^2-1)^n\}]_{-1}^{1}$$
$$-\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} D\big[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}\big]D^{n+s-1}\{(x^2-1)^n\}dx.$$

The first term vanishes at both limits and we have

$$\int_{-1}^{1}\{P_n^s(x)\}^2dx = \frac{-1}{2^{2n}(n!)^2}\int_{-1}^{1} D[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}]D^{n+s-1}\{(x^2-1)^n\}dx. \tag{7.35}$$

We can continue to integrate Eq. (7.35) by parts and the first term continues to vanish since $D^p[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}]$ contains the factor $(1-x^2)$ when $p < s$

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and $D^{n+s-p}\{(x^2-1)^n\}$ contains it when $p \geq s$. After integrating $(n+s)$ times we find

$$\int_{-1}^{1} \{P_n^s(x)\}^2 dx = \frac{(-1)^{n+s}}{2^{2n}(n!)^2} \int_{-1}^{1} D^{n+s}[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}](x^2-1)^n dx. \quad (7.36)$$

But $D^{n+s}\{(x^2-1)^n\}$ is a polynomial of degree $(n-s)$ so that $(1-x^2)^s D^{n+s}\{(x^2-1)^n\}$ is of degree $n-2+2s = n+s$. Hence the first factor in the integrand is a polynomial of degree zero. We can find this constant by examining the following:

$$D^{n+s}(x^{2n}) = 2n(2n-1)(2n-2)\cdots(n-+1)x^{n-s}.$$

Hence the highest power in $(1-x^2)^s D^{n+s}\{(x^2-1)^n\}$ is the term

$$(-1)^s 2n(2n-1)\cdots(n-s+1)x^{n+s},$$

so that

$$D^{n+s}[(1-x^2)^s D^{n+s}\{(x^2-1)^n\}] = (-1)^s(2n)!\,\frac{(n+s)!}{(n-s)!}.$$

Now Eq. (7.36) gives, by writing $x = \cos\theta$,

$$\int_{-1}^{1} P_n^s\{(x)\}^2 dx = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (2n)!\,\frac{(n+s)!}{(n-s)!}(x^2-1)^n dx$$
$$= \frac{2}{2n+1}\frac{(n+s)!}{(n-s)!} \quad (7.37)$$

Hermite's equation

Hermite's equation is

$$y'' - 2xy' + 2\nu y = 0, \quad (7.38)$$

where $y' = dy/dx$. The reader will see this equation in quantum mechanics (when solving the Schrödinger equation for a linear harmonic potential function).

The origin $x = 0$ is an ordinary point and we may write the solution in the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{j=0}^{\infty} a_j x^j. \quad (7.39)$$

Differentiating the series term by term, we have

$$y' = \sum_{j=0}^{\infty} j a_j x^{j-1}, \qquad y'' = \sum_{j=0}^{\infty} (j+1)(j+2)a_{j+2}x^j.$$

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Substituting these into Eq. (7.38) we obtain

$$\sum_{j=0}^{\infty} \left[(j+1)(j+2)a_{j+2} + 2(\nu - j)a_j\right]x^j = 0.$$

For a power series to vanish the coefficient of each power of x must be zero; this gives

$$(j+1)(j+2)a_{j+2} + 2(\nu - j)a_j = 0,$$

from which we obtain the recurrence relations

$$a_{j+2} = \frac{2(j-\nu)}{(j+1)(j+2)} a_j. \tag{7.40}$$

We obtain polynomial solutions of Eq. (7.38) when $\nu = n$, a positive integer. Then Eq. (7.40) gives

$$a_{n+2} = a_{n+4} = \cdots = 0.$$

For even n, Eq. (7.40) gives

$$a_2 = (-1)\frac{2n}{2!}a_0, \quad a_4 = (-1)^2\frac{2^2(n-2)n}{4!}a_0, \quad a_6 = (-1)^3\frac{2^3(n-4)(n-2)n}{6!}a_0$$

and generally

$$a_n = (-1)^{n/2}\frac{2^{n/2}n(n-2)\cdots 4\times 2}{n!}a_0.$$

This solution is called a Hermite polynomial of degree n and is written $H_n(x)$. If we choose

$$a_0 = \frac{(-1)^{n/2}2^{n/2}n!}{n(n-2)\cdots 4\times 2} = \frac{(-1)^{n/2}n!}{(n/2)!}$$

we can write

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \cdots. \tag{7.41}$$

When n is odd the polynomial solution of Eq. (7.38) can still be written as Eq. (7.41) if we write

$$a_1 = \frac{(-1)^{(n-1)/2}2n!}{(n/2-1/2)!}.$$

In particular,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_3(x) = 4x^2 - 2, \quad H_3(x) = 8x^2 - 12x,$$
$$H_4(x) = 16x^4 - 48x^2 + 12, \quad H_5(x) = 32x^5 - 160x^3 + 120x, \ldots .$$

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Rodrigues' formula for Hermite polynomials $H_n(x)$

The Hermite polynomials are also given by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{7.42}$$

To prove this formula, let us write $q = e^{-x^2}$. Then

$$Dq + 2xq = 0, \quad D = \frac{d}{dx}.$$

Differentiate this $(n+1)$ times by the Leibnitz' rule giving

$$D^{n+2}q + 2xD^{n+1}q + 2(n+1)D^n q = 0.$$

Writing $y = (-1)^n D^n q$ gives

$$D^2 y + 2xDy + 2(n+1)y = 0 \tag{7.43}$$

substitute $u = e^{x^2} y$ then

$$Du = e^{x^2}\{2xy + Dy\}$$

and

$$D^2 u = e^{x^2}\{D^2 y + 4xDy + 4x^2 y + 2y\}.$$

Hence by Eq. (7.43) we get

$$D^2 u - 2xDu + 2nu = 0,$$

which indicates that

$$u = (-1)^n e^{x^2} D^n (e^{-x^2})$$

is a polynomial solution of Hermite's equation (7.38).

Recurrence relations for Hermite polynomials

Rodrigues' formula gives on differentiation

$$H_n'(x) = (-1)^n 2xe^{x^2} D^n(e^{-x^2}) + (-1)^n e^{x^2} D^{n+1}(e^{-x^2}).$$

that is,

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x). \tag{7.44}$$

Eq. (7.44) gives on differentiation

$$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x).$$

Now $H_n(x)$ satisfies Hermite's equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

Eliminating $H_n''(x)$ from the last two equations, we obtain

$$2xH_n'(x) - 2nH_n(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x)$$

which reduces to

$$H_{n+1}'(x) = 2(n+1)H_n(x). \tag{7.45}$$

Replacing n by $n+1$ in Eq. (7.44), we have

$$H_{n+1}'(x) = 2xH_{n+1}(x) - H_{n+2}(x).$$

Combining this with Eq. (7.45) we obtain

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x). \tag{7.46}$$

This will quickly give the higher polynomials.

Generating function for the $H_n(x)$

By using Rodrigues' formula we can also find a generating formula for the $H_n(x)$. This is

$$\Phi(x,t) = e^{2tx-t^2} = e^{\{x^2-(t-x)^2\}} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{7.47}$$

Differentiating Eq. (7.47) n times with respect to t we get

$$e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2} = e^{x^2}(-1)^n \frac{\partial^n}{\partial x^n} e^{-(t-x)^2} = \sum_{k=0}^{\infty} H_{n+k}(x) \frac{t^k}{k!}.$$

Put $t = 0$ in the last equation and we obtain Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

The orthogonal Hermite functions

These are defined by

$$F_n(x) = e^{-x^2/2} H_n(x); \tag{7.48}$$

from which we have

$$DF_n(x) = -xF_n(x) + e^{-x^2/2} H_n'(x),$$

$$D^2F_n(x) = e^{-x^2/2} H_n''(x) - 2xe^{-x^2/2} H_n'(x) + x^2 e^{-x^2/2} H_n(x) - F_n(x)$$

$$= e^{-x^2/2}[H_n''(x) - 2xH_n'(x)] + x^2F_n(x) - F_n(x),$$

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but $H_n''(x) - 2xH_n'(x) = -2nH_n(x)$, so we can rewrite the last equation as

$$D^2F_n(x) = e^{-x^2/2}[-2nH_n'(x)] + x^2F_n(x) - F_n(x)$$
$$= -2nF_n(x) + x^2F_n(x) - F_n(x),$$

which gives

$$D^2F_n(x) - x^2F_n(x) + (2n+1)F_n(x) = 0. \tag{7.49}$$

We can now show that the set $\{F_n(x)\}$ is orthogonal in the infinite range $-\infty < x < \infty$. Multiplying Eq. (7.49) by $F_m(x)$ we have

$$F_m(x)D^2F_n(x) - x^2F_n(x)F_m(x) + (2n+1)F_n(x)F_m(x) = 0.$$

Interchanging m and n gives

$$F_n(x)D^2F_m(x) - x^2F_m(x)F_n(x) + (2m+1)F_m(x)F_n(x) = 0.$$

Subtracting the last two equations from the previous one and then integrating from $-\infty$ to $+\infty$, we have

$$I_{n,m} = \int_{-\infty}^{\infty} F_n(x)F_m(x)dx = \frac{1}{2(n-m)}\int_{-\infty}^{\infty}(F_n''F_m - F_m''F_n)dx.$$

The integration by parts gives

$$2(n-m)I_{n,m} = \left[F_n'F_m - F_m'F_n\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(F_n'F_m' - F_m'F_n')dx.$$

Since the right hand side vanishes at both limits and if $m \neq m$, we have

$$I_{n,m} = \int_{-\infty}^{\infty} F_n(x)F_m(x)dx = 0. \tag{7.50}$$

When $n = m$ we can proceed as follows

$$I_{n,n} = \int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_n(x)dx = \int_{-\infty}^{\infty} e^{x^2}D^n(e^{-x^2})D^m(e^{-x^2})dx.$$

Integration by parts, that is, $\int udv = uv - \int vdu$ with $u = e^{-x^2}D^n(e^{-x^2})$ and $v = D^{n-1}(e^{-x^2})$, gives

$$I_{n,n} = -\int_{-\infty}^{\infty}[2xe^{x^2}D^n(e^{-x^2}) + e^{x^2}D^{n+1}(e^{-x^2})]D^{n-1}(e^{-x^2})dx.$$

By using Eq. (7.43) which is true for $y = (-1)^nD^nq = (-1)^nD^n(e^{-x^2})$ we obtain

$$I_{n,n} = \int_{-\infty}^{\infty} 2ne^{x^2}D^{n-1}(e^{-x^2})D^{n-1}(e^{-x^2})dx = 2nI_{n-1,n-1}.$$

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Since

$$I_{0,0} = \int_{-\infty}^{\infty} e^{-x^2} dx = \Gamma(1/2) = \sqrt{\pi},$$

we find that

$$I_{n,n} = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_n(x) dx = 2^n n! \sqrt{\pi}. \tag{7.51}$$

We can also use the generating function for the Hermite polynomials:

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}, \quad e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{H_m(x)s^m}{m!}.$$

Multiplying these, we have

$$e^{2tx-t^2+2sx-s^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)s^m t^n}{m!n!}.$$

Multiplying by e^{-x^2} and integrating from $-\infty$ to ∞ gives

$$\int_{-\infty}^{\infty} e^{-[(x+s+t)^2-2st]} dx = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx.$$

Now the left hand side is equal to

$$e^{2st} \int_{-\infty}^{\infty} e^{-(x+s+t)^2} dx = e^{2st} \int_{-\infty}^{\infty} e^{-u^2} du = e^{2st}\sqrt{\pi} = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{2^m s^m t^m}{m!}.$$

By equating coefficients the required result follows.

It follows that the functions $(1/2^n n! \sqrt{n})^{1/2} e^{-x^2} H_n(x)$ form an orthonormal set. We shall assume it is complete.

Laguerre's equation

Laguerre's equation is

$$xD^2 y + (1-x)Dy + \nu y = 0. \tag{7.52}$$

This equation and its solutions (Laguerre functions) are of interest in quantum mechanics (e.g., the hydrogen problem). The origin $x = 0$ is a regular singular point and so we write

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\rho}. \tag{7.53}$$

By substitution, Eq. (7.52) becomes

$$\sum_{k=0}^{\infty} [(k+\rho)^2 a_k x^{k+\rho-1} + (\nu - k + \rho) a_k x^k] = 0 \tag{7.54}$$

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from which we find that the indicial equation is $\rho^2 = 0$. And then (7.54) reduces to

$$\sum_{k=0}^{\infty} [k^2 a_k x^{k-1} + (\nu - k) a_k x^k] = 0.$$

Changing $k-1$ to k' in the first term, then renaming $k' = k$, we obtain

$$\sum_{k=0}^{\infty} \{(k+1)^2 a_{k+1} + (\nu - k) a_k\} x^k = 0,$$

whence the recurrence relations are

$$a_{k+1} = \frac{k-\nu}{(k+1)^2} a_k. \tag{7.55}$$

When ν is a positive integer n, the recurrence relations give $a_{k+1} = a_{k+2} = \cdots = 0$, and

$$a_1 = \frac{-n}{1^2} a_0, \qquad a_2 = \frac{-(n-1)}{2^2} a_1 = \frac{(-1)^2 (n-1)n}{(1\times 2)^2} a_0,$$

$$a_3 = \frac{-(n-2)}{3^2} a_2 = \frac{(-1)^3 (n-2)(n-1)n}{(1\times 2\times 3)^2} a_0, \quad \text{etc.}$$

In general

$$a_k = (-1)^k \frac{(n-k+1)(n-k+2)\cdots(n-1)n}{(k!)^2} a_0. \tag{7.56}$$

We usually choose $a_0 = (-1)n!$, then the polynomial solution of Eq. (7.52) is given by

$$L_n(x) = (-1)^n \left\{ x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2 (n-1)^2}{2!} x^{n-2} - + \cdots + (-1)^n n! \right\}. \tag{7.57}$$

This is called the Laguerre polynomial of degree n. We list the first four Laguerre polynomials below:

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = 2 - 4x + x^2, \quad L_3(x) = 6 - 18x + 9x^2 - x^3.$$

The generating function for the Laguerre polynomials $L_n(x)$

This is given by

$$\Phi(x,z) = \frac{e^{-xz/(1-z)}}{1-z} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n. \tag{7.58}$$

By writing the series for the exponential and collecting powers of z, you can verify the first few terms of the series. And it is also straightforward to show that

$$x\frac{\partial^2\Phi}{\partial x^2} + (1-x)\frac{\partial\Phi}{\partial x} + z\frac{\partial\Phi}{\partial z} = 0.$$

Substituting the right hand side of Eq. (7.58), that is, $\Phi(x,z) = \sum_{n=0}^{\infty}[L_n(x)/n!]z^n$, into the last equation we see that the functions $L_n(x)$ satisfy Laguerre's equation. Thus we identify $\Phi(x,z)$ as the generating function for the Laguerre polynomials.

Now multiplying Eq. (7.58) by z^{-n-1} and integrating around the origin, we obtain

$$L_n(x) = \frac{n!}{2\pi i}\oint \frac{e^{-xz/(1-z)}}{(1-z)z^{n+1}}dz, \tag{7.59}$$

which is an integral representation of $L_n(x)$.

By differentiating the generating function in Eq. (7.58) with respect to x and z, we obtain the recurrence relations

$$\left.\begin{aligned} L_{n+1}(x) &= (2n+1-x)L_n(x) - n^2L_{n-1}(x),\\ nL_{n-1}(x) &= nL'_{n-1}(x) - L'_n(x). \end{aligned}\right\} \tag{7.60}$$

Rodrigues' formula for the Laguerre polynomials $L_n(x)$

The Laguerre polynomials are also given by Rodrigues' formula

$$L_n(x) = e^x\frac{d^n}{dx^n}(x^ne^{-x}). \tag{7.61}$$

To prove this formula, let us go back to the integral representation of $L_n(x)$, Eq. (7.59). With the transformation

$$\frac{xz}{1-z} = s - x \quad \text{or} \quad z = \frac{s-x}{s},$$

Eq. (7.59) becomes

$$L_n(x) = \frac{n!e^x}{2\pi i}\oint \frac{s^ne^{-n}}{(s-x)^{n+1}}ds,$$

the new contour enclosing the point $s = x$ in the s plane. By Cauchy's integral formula (for derivatives) this reduces to

$$L_n(x) = e^x\frac{d^n}{dx^n}(x^ne^{-x}),$$

which is Rodrigues' formula.

Alternatively, we can differentiate Eq. (7.58) n times with respect to z and afterwards put $z=0$, and thus obtain

$$e^x\lim_{z\to 0}\frac{\partial^n}{\partial z^n}\left[(1-z)^{-1}\exp\left(\frac{-x}{1-z}\right)\right] = L_n(x).$$

But

$$\lim_{z\to 0}\frac{\partial^n}{\partial z^n}\left[(1-z)^{-1}\exp\left(\frac{-x}{1-z}\right)\right]=\frac{d^n}{dx^n}(x^n e^{-x}),$$

hence

$$L_n(x)=e^x\frac{d^n}{dx^n}(x^n e^{-x}).$$

The orthogonal Laguerre functions

The Laguerre polynomials, $L_n(x)$, do not by themselves form an orthogonal set. But the functions $e^{-x/2}L_n(x)$ are orthogonal in the interval $(0, \infty)$. For any two Laguerre polynomials $L_m(x)$ and $L_n(x)$ we have, from Laguerre's equation,

$$xL_m''+(1-x)L_m'+mL_m=0,$$
$$xL_n''+(1-x)L_n'+mL_n=0.$$

Multiplying these equations by $L_n(x)$ and $L_m(x)$ respectively and subtracting, we find

$$x[L_nL_m''-L_mL_n'']+(1-x)[L_nL_m'-L_mL_n']=(n-m)L_mL_n$$

or

$$\frac{d}{dx}[L_nL_m'-L_mL_n']+\frac{1-x}{x}[L_nL_m'-L_mL_n']=\frac{(n-m)L_mL_n}{x}.$$

Then multiplying by the integrating factor

$$\exp\int[(1-x)/x]dx=\exp(\ln x-x)=xe^{-x},$$

we have

$$\frac{d}{dx}\{xe^{-x}[L_nL_m'-L_mL_n']\}=(n-m)e^{-x}L_mL_n.$$

Integrating from 0 to ∞ gives

$$(n-m)\int_0^\infty e^{-x}L_m(x)L_n(x)dx=xe^{-x}[L_nL_m'-L_mL_n']|_0^\infty=0.$$

Thus if $m\neq n$

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx=0\quad(m\neq n), \tag{7.62}$$

which proves the required result.

Alternatively, we can use Rodrigues' formula (7.61). If m is a positive integer,

$$\int_0^\infty e^{-x}x^mL_m(x)dx=\int_0^\infty x^m\frac{d^n}{dx^n}(x^ne^{-x})dx=(-1)^m m!\int_0^\infty\frac{d^{n-m}}{dx^{n-m}}(x^ne^{-x})dx, \tag{7.63}$$

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the last step resulting from integrating by parts m times. The integral on the right hand side is zero when $n > m$ and, since $L_n(x)$ is a polynomial of degree m in x, it follows that

$$\int_0^\infty e^{-x} L_m(x) L_n(x) dx = 0 \quad (m \neq n),$$

which is Eq. (7.62). The reader can also apply Eq. (7.63) to show that

$$\int_0^\infty e^{-x} \{L_n(x)\}^2 dx = (n!)^2. \qquad (7.64)$$

Hence the functions $\{e^{-x/2} L_n(x)/n!\}$ form an orthonormal system.

The associated Laguerre polynomials $L_n^m(x)$

Differentiating Laguerre's equation (7.52) m times by the Leibnitz theorem we obtain

$$xD^{m+2}y + (m+1-x)D^{m+1}y + (n-m)D^m y = 0 \quad (\nu = n)$$

and writing $z = D^m y$ we obtain

$$xD^2 z + (m+1-x)Dz + (n-m)z = 0. \qquad (7.65)$$

This is Laguerre's associated equation and it clearly possesses a polynomial solution

$$z = D^m L_n(x) \equiv L_n^m(x) \quad (m \le n), \qquad (7.66)$$

called the associated Laguerre polynomial of degree $(n-m)$. Using Rodrigues' formula for Laguerre polynomial $L_n(x)$, Eq. (7.61), we obtain

$$L_n^m(x) = \frac{d^m}{dx^m} L_n(x) = \frac{d^m}{dx^m}\left\{ e^x \frac{d^n}{dx^n}(x^n e^{-x}) \right\}. \qquad (7.67)$$

This result is very useful in establishing further properties of the associated Laguerre polynomials. The first few polynomials are listed below:

$$L_0^0(x) = 1; \quad L_1^0(x) = 1 - x; \quad L_1^1(x) = -1;$$

$$L_2^0(x) = 2 - 4x + x^2; \quad L_2^1(x) = -4 + 2x; \quad L_2^2(x) = 2.$$

Generating function for the associated Laguerre polynomials

The Laguerre polynomial $L_n(x)$ can be generated by the function

$$\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^\infty L_n(x)\frac{t^n}{n!}.$$

Differentiating this k times with respect to x, it is seen at once that

$$(-1)^k(1-t)^{-1}\left(\frac{t}{1-t}\right)^k \exp\left(\frac{-xt}{1-t}\right) = \sum_{\lambda=k}^{\infty} \frac{L_\lambda^k(x)}{\lambda!} t^\lambda. \tag{7.68}$$

Associated Laguerre function of integral order

A function of great importance in quantum mechanics is the associated Laguerre function that is defined as

$$G_n^m(x) = e^{-x/2} x^{(m-1)/2} L_n^m(x) \quad (m \le n). \tag{7.69}$$

It is significant largely because $|G_n^m(x)| \to 0$ as $x \to \infty$. It satisfies the differential equation

$$x^2D^2u + 2xDu + \left[\left(n - \frac{m-1}{2}\right)x - \frac{x^2}{4} - \frac{m^2-1}{4}\right]u = 0. \tag{7.70}$$

If we substitute $u = e^{-x/2}x^{(m-1)/2}z$ in this equation, it reduces to Laguerre's associated equation (7.65). Thus $u = G_n^m$ satisfies Eq. (7.70). You will meet this equation in quantum mechanics in the study of the hydrogen atom.

Certain integrals involving G_n^m are often used in quantum mechanics and they are of the form

$$I_{n,m} = \int_0^\infty e^{-x}x^{k-1}L_n^k(x)L_m^k(x)x^p dx,$$

where p is also an integer. We will not consider these here and instead refer the interested reader to the following book: *The Mathematics of Physics and Chemistry*, by Henry Margenau and George M. Murphy; D. Van Nostrand Co. Inc., New York, 1956.

Bessel's equation

The differential equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{7.71}$$

in which α is a real and positive constant, is known as Bessel's equation and its solutions are called Bessel functions. These functions were used by Bessel (Friedrich Wilhelm Bessel, 1784–1864, German mathematician and astronomer) extensively in a problem of dynamical astronomy. The importance of this equation and its solutions (Bessel functions) lies in the fact that they occur frequently in the boundary-value problems of mathematical physics and engineering involving cylindrical symmetry (so Bessel functions are sometimes called cylindrical functions), and many others. There are whole books on Bessel functions.

The origin is a regular singular point, and all other values of x are ordinary points. At the origin we seek a series solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+\rho} \quad (a_0 \neq 0). \tag{7.72}$$

Substituting this and its derivatives into Bessel's equation (7.71), we have

$$\sum_{m=0}^{\infty}(m+\rho)(m+\rho-1)a_m x^{m+\rho} + \sum_{m=0}^{\infty}(m+\rho)a_m x^{m+\rho} + \sum_{m=0}^{\infty} a_m x^{m+\rho+2} - \alpha^2 \sum_{m=0}^{\infty} a_m x^{m+\rho} = 0.$$

This will be an identity if and only if the coefficient of every power of x is zero. By equating the sum of the coefficients of $x^{k+\rho}$ to zero we find

$$\rho(\rho-1)a_0 + \rho a_0 - \alpha^2 a_0 = 0 \qquad (k=0), \tag{7.73a}$$

$$(\rho-1)\rho a_1 + (\rho+1)a_1 - \alpha^2 a_1 = 0 \qquad (k=1), \tag{7.73b}$$

$$(k+\rho)(k+\rho-1)a_k + (k+\rho)a_k + a_{k-2} - \alpha^2 a_k = 0 \qquad (k=2,3,\ldots). \tag{7.73c}$$

From Eq. (7.73a) we obtain the indicial equation

$$\rho(\rho-1) + \rho - \alpha^2 = (\rho+\alpha)(\rho-\alpha) = 0.$$

The roots are $\rho = \pm\alpha$. We first determine a solution corresponding to the positive root. For $\rho = +\alpha$, Eq. (7.73b) yields $a_1 = 0$, and Eq. (7.73c) takes the form

$$(k+2\alpha)ka_k + a_{k-2} = 0, \qquad \text{or} \qquad a_k = \frac{-1}{k(k+2\alpha)}a_{k-2}, \tag{7.74}$$

which is a recurrence formula: since $a_1 = 0$ and $\alpha \geq 0$, it follows that $a_3 = 0, a_5 = 0, \ldots,$ successively. If we set $k = 2m$ in Eq. (7.74), the recurrence formula becomes

$$a_{2m} = -\frac{1}{2^2 m(\alpha+m)}a_{2m-2}, \qquad m = 1, 2, \ldots \tag{7.75}$$

and we can determine the coefficients a_2, a_4, successively. We can rewrite a_{2m} in terms of a_0:

$$a_{2m} = \frac{(-1)^m}{2^{2m} m!(\alpha+m)\cdots(\alpha+2)(\alpha+1)}a_0.$$

Now a_{2m} is the coefficient of $x^{\alpha+2m}$ in the series (7.72) for y. Hence it would be convenient if a_{2m} contained the factor $2^{\alpha+2m}$ in its denominator instead of just 2^{2m}. To achieve this, we write

$$a_{2m} = \frac{(-1)^m}{2^{\alpha+2m} m!(\alpha+m)\cdots(\alpha+2)(\alpha+1)}(2^\alpha a_0).$$

Furthermore, the factors

$$(\alpha+m)\cdots(\alpha+2)(\alpha+1)$$

suggest a factorial. In fact, if α were an integer, a factorial could be created by multiplying numerator by $\alpha!$. However, since α is not necessarily an integer, we must use not $\alpha!$ but its generalization $\Gamma(\alpha+1)$ for this purpose. Then, except for the values

$$\alpha = -1, -2, -3, \ldots$$

for which $\Gamma(\alpha+1)$ is not defined, we can write

$$a_{2m} = \frac{(-1)^m}{2^{\alpha+2m} m!(\alpha+m)\cdots(\alpha+2)(\alpha+1)\Gamma(\alpha+1)}[2^\alpha \Gamma(\alpha+1) a_0].$$

Since the gamma function satisfies the recurrence relation $z\Gamma(z) = \Gamma(z+1)$, the expression for a_{2m} becomes finally

$$a_{2m} = \frac{(-1)^m}{2^{\alpha+2m} m!\Gamma(\alpha+m+1)}[2^\alpha \Gamma(\alpha+1) a_0].$$

Since a_0 is arbitrary, and since we are looking only for particular solutions, we choose

$$a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)},$$

so that

$$a_{2m} = \frac{(-1)^m}{2^{\alpha+2m} m!\Gamma(\alpha+m+1)}, \qquad a_{2m+1} = 0$$

and the series for y is, from Eq. (7.72),

$$\begin{aligned} y(x) &= x^\alpha \left[\frac{1}{2^\alpha \Gamma(\alpha+1)} - \frac{x^2}{2^{\alpha+2}\Gamma(\alpha+2)} + \frac{x^4}{2^{\alpha+4} 2!\Gamma(\alpha+3)} - + \cdots\right] \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\alpha+2m} m!\Gamma(\alpha+m+1)} x^{\alpha+2m}. \end{aligned} \tag{7.76}$$

The function defined by this infinite series is known as the Bessel function of the first kind of order α and is denoted by the symbol $J_\alpha(x)$. Since Bessel's equation

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of order α has no finite singular points except the origin, the ratio test will show that the series for $J_\alpha(x)$ converges for all values of x if $\alpha \geq 0$.

When $\alpha = n$, an integer, solution (7.76) becomes, for $n \geq 0$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}. \tag{7.76a}$$

The graphs of $J_0(x), J_1(x)$, and $J_2(x)$ are shown in Fig. 7.3. Their resemblance to the graphs of cos x and sin x is interesting (Problem 7.16 illustrates this for the first few terms). Fig. 7.3 also illustrates the important fact that for every value of α the equation $J_\alpha(x) = 0$ has infinitely many real roots.

With the second root $\rho = -\alpha$ of the indicial equation, the recurrence relation takes the form (from Eq. (7.73c))

$$a_k = \frac{-1}{k(k-2\alpha)} a_{k-2}. \tag{7.77}$$

If α is not an integer, this leads to an independent second solution that can be written

$$J_{-\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-\alpha+m+1)} (x/2)^{-\alpha+2m} \tag{7.78}$$

and the complete solution of Bessel's equation is then

$$y(x) = AJ_\alpha(x) + BJ_{-\alpha}(x), \tag{7.79}$$

where A and B are arbitrary constants.

When α is a positive integer n, it can be shown that the formal expression for $J_{-n}(x)$ is equal to $(-1)^n J_n(x)$. So $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and Eq. (7.79) cannot be a general solution. In fact, if α is a positive integer, the recurrence

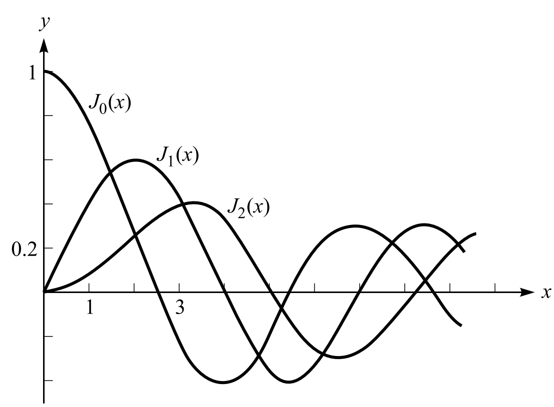

Figure 7.3. Bessel functions of the first kind.

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relation (7.77) breaks down when $2\alpha = k$ and a second solution has to be found by other methods. There is a difficulty also when $\alpha = 0$, in which case the two roots of the indicial equation are equal; the second solution must also found by other methods. These will be discussed in next section.

The results of Problem 7.16 are a special case of an important general theorem which states that $J_\alpha(x)$ is expressible in finite terms by means of algebraic and trigonometrical functions of x whenever α is half of an odd integer. Further examples are

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right),$$

$$J_{-5/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left\{\frac{3\sin x}{x} + \left(\frac{3}{x^2} - 1\right)\cos x\right\}.$$

The functions $J_{(n+1/2)}(x)$ and $J_{-(n+1/2)}(x)$, where n is a positive integer or zero, are called spherical Bessel functions; they have important applications in problems of wave motion in which spherical polar coordinates are appropriate.

Bessel functions of the second kind $Y_n(x)$

For integer $\alpha = n, J_n(x)$ and $J_{-n}(x)$ are linearly dependent and do not form a fundamental system. We shall now obtain a second independent solution, starting with the case $n = 0$. In this case Bessel's equation may be written

$$xy'' + y' + xy = 0, \tag{7.80}$$

the indicial equation (7.73a) now, with $\alpha = 0$, has the double root $\rho = 0$. Then we see from Eq. (7.33) that the desired solution must be of the form

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m. \tag{7.81}$$

Next we substitute y_2 and its derivatives

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1},$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

into Eq. (7.80). Then the logarithmic terms disappear because J_0 is a solution of Eq. (7.80), the other two terms containing J_0 cancel, and we find

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

From Eq. (7.76a) we obtain J_0' as

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}.$$

By inserting this series we have

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

We first show that A_m with odd subscripts are all zero. The coefficient of the power x^0 is A_1 and so $A_1 = 0$. By equating the sum of the coefficients of the power x^{2s} to zero we obtain

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 1, 2, \ldots .$$

Since $A_1 = 0$, we thus obtain $A_3 = 0, A_5 = 0, \ldots$, successively. We now equate the sum of the coefficients of x^{2s+1} to zero. For $s = 0$ this gives

$$-1 + 4A_2 = 0 \qquad \text{or} \qquad A_2 = 1/4.$$

For the other values of s we obtain

$$\frac{(-1)^{s+1}}{2^s(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0.$$

For $s = 1$ this yields

$$1/8 + 16A_4 + A_2 = 0 \quad \text{or} \quad A_4 = -3/128$$

and in general

$$A_{2m} = \frac{(-1)^{m-1}}{2^m(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \qquad m = 1, 2, \ldots . \tag{7.82}$$

Using the short notation

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

and inserting Eq. (7.82) and $A_1 = A_3 = \cdots = 0$ into Eq. (7.81) we obtain the result

$$\begin{aligned} y_2(x) &= J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \\ &= J_0(x) \ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + - \cdots . \end{aligned} \tag{7.83}$$

Since J_0 and y_2 are linearly independent functions, they form a fundamental system of Eq. (7.80). Of course, another fundamental system is obtained by replacing y_2 by an independent particular solution of the form $a(y_2 + bJ_0)$, where $a(\neq 0)$ and b are constants. It is customary to choose $a = 2/\pi$ and

$b = \gamma - \ln 2$, where $\gamma = 0.577\,215\,664\,90\ldots$ is the so-called Euler constant, which is defined as the limit of

$$1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

as s approaches infinity. The standard particular solution thus obtained is known as the Bessel function of the second kind of order zero or Neumann's function of order zero and is denoted by $Y_0(x)$:

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]. \tag{7.84}$$

If $\alpha = 1, 2, \ldots$, a second solution can be obtained by similar manipulations, starting from Eq. (7.35). It turns out that in this case also the solution contains a logarithmic term. So the second solution is unbounded near the origin and is useful in applications only for $x \neq 0$.

Note that the second solution is defined differently, depending on whether the order α is integral or not. To provide uniformity of formalism and numerical tabulation, it is desirable to adopt a form of the second solution that is valid for all values of the order. The common choice for the standard second solution defined for all α is given by the formula

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}, \quad Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x). \tag{7.85}$$

This function is known as the Bessel function of the second kind of order α. It is also known as Neumann's function of order α and is denoted by $N_\alpha(x)$ (Carl Neumann 1832–1925, German mathematician and physicist). In G. N. Watson's *A Treatise on the Theory of Bessel Functions* (2nd ed. Cambridge University Press, Cambridge, 1944), it was called Weber's function and the notation $Y_\alpha(x)$ was used. It can be shown that

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

We plot the first three $Y_n(x)$ in Fig. 7.4.

A general solution of Bessel's equation for all values of α can now be written:

$$y(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x).$$

In some applications it is convenient to use solutions of Bessel's equation that are complex for all values of x, so the following solutions were introduced

$$\left.\begin{aligned} H_\alpha^{(1)}(x) &= J_\alpha(x) + iY_\alpha(x), \\ H_\alpha^{(2)}(x) &= J_\alpha(x) - iY_\alpha(x). \end{aligned}\right\} \tag{7.86}$$

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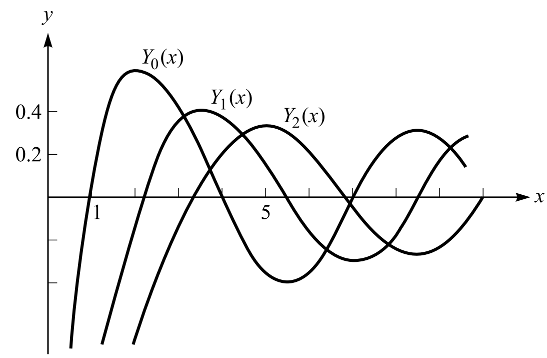

Figure 7.4. Bessel functions of the second kind.

These linearly independent functions are known as Bessel functions of the third kind of order α or first and second Hankel functions of order α (Hermann Hankel, 1839–1873, German mathematician).

To illustrate how Bessel functions enter into the analysis of physical problems, we consider one example in classical physics: small oscillations of a hanging chain, which was first considered as early as 1732 by Daniel Bernoulli.

Hanging flexible chain

Fig. 7.5 shows a uniform heavy flexible chain of length l hanging vertically under its own weight. The x-axis is the position of stable equilibrium of the chain and its lowest end is at $x = 0$. We consider the problem of small oscillations in the vertical xy plane caused by small displacements from the stable equilibrium position. This is essentially the problem of the vibrating string which we discussed in Chapter 4, with two important differences: here, instead of being constant, the tension T at a given point of the chain is equal to the weight of the chain below that point, and now one end of the chain is free, whereas before both ends were fixed. The analysis of Chapter 4 generally holds. To derive an equation for y, consider an element dx, then Newton's second law gives

$$\left(T\frac{\partial y}{\partial x}\right)_2 - \left(T\frac{\partial y}{\partial x}\right)_1 = \rho dx\frac{\partial^2 y}{\partial t^2}$$

or

$$\rho dx\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx,$$

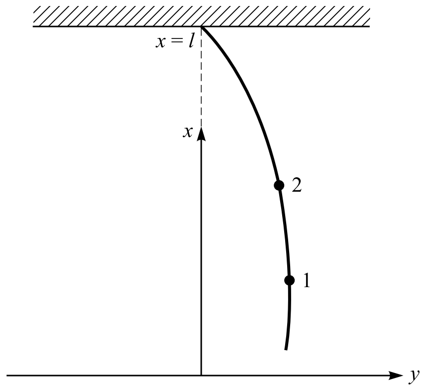

Figure 7.5. A flexible chain.

from which we obtain

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right).$$

Now $T = \rho g x$. Substituting this into the above equation for y, we obtain

$$\frac{\partial^2 y}{\partial t^2} = g\frac{\partial y}{\partial x} + gx\frac{\partial^2 y}{\partial x^2},$$

where y is a function of two variables x and t. The first step in the solution is to separate the variables. Let us attempt a solution of the form $y(x, t) = u(x)f(t)$. Substitution of this into the partial differential equation yields two equations:

$$f''(t) + \omega^2 f(t) = 0, \quad xu''(x) + u'(x) + (\omega^2/g)u(x) = 0,$$

where ω^2 is the separation constant. The differential equation for $f(t)$ is ready for integration and the result is $f(t) = \cos(\omega t - \delta)$, with δ a phase constant. The differential equation for $u(x)$ is not in a recognizable form yet. To solve it, first change variables by putting

$$x = gz^2/4, \quad w(z) = u(x),$$

then the differential equation for $u(x)$ becomes Bessel's equation of order zero:

$$zw''(z) + w'(z) + \omega^2 zw(z) = 0.$$

Its general solution is

$$w(z) = AJ_0(\omega z) + BY_0(\omega z)$$

or

$$u(x) = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) + BY_0\left(2\omega\sqrt{\frac{x}{g}}\right).$$

Since $Y_0(2\omega\sqrt{x/g}) \to -\infty$ as $x \to 0$, we are forced by physics to choose $B = 0$ and then

$$y(x,t) = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right)\cos(\omega t - \delta).$$

The upper end of the chain at $x = l$ is fixed, requiring that

$$J_0\left(2\omega\sqrt{\frac{\ell}{g}}\right) = 0.$$

The frequencies of the normal vibrations of the chain are given by

$$2\omega_n\sqrt{\frac{\ell}{g}} = \alpha_n,$$

where α_n are the roots of J_0. Some values of $J_0(x)$ and $J_1(x)$ are tabulated at the end of this chapter.

Generating function for $J_n(x)$

The function

$$\Phi(x,t) = e^{(x/2)(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{7.87}$$

is called the generating function for Bessel functions of the first kind of integral order. It is very useful in obtaining properties of $J_n(x)$ for integral values of n which can then often be proved for all values of n.

To prove Eq. (7.87), let us consider the exponential functions $e^{xt/2}$ and $e^{-xt/2}$. The Laurent expansions for these two exponential functions about $t = 0$ are

$$e^{xt/2} = \sum_{k=0}^{\infty}\frac{(xt/2)^k}{k!},\ e^{-xt/2} = \sum_{m=0}^{\infty}\frac{(-xt/2)^k}{m!}.$$

Multiplying them together, we get

$$e^{x(t-t^{-1})/2} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{k!m!}\left(\frac{x}{2}\right)^{k+m} t^{k-m}. \tag{7.88}$$

It is easy to recognize that the coefficient of the t^0 term which is made up of those terms with $k = m$ is just $J_0(x)$:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}x^{2k} = J_0(x).$$

Similarly, the coefficient of the term t^n which is made up of those terms for which $k - m = n$ is just $J_n(x)$:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)!k!2^{2k+n}} x^{2k+n} = J_n(x).$$

This shows clearly that the coefficients in the Laurent expansion (7.88) of the generating function are just the Bessel functions of integral order. Thus we have proved Eq. (7.87).

Bessel's integral representation

With the help of the generating function, we can express $J_n(x)$ in terms of a definite integral with a parameter. To do this, let $t = e^{i\theta}$ in the generating function, then

$$\begin{aligned} e^{x(t-t^{-1})/2} &= e^{x(e^{i\theta}-e^{-i\theta})/2} = e^{ix\sin\theta} \\ &= \cos(x\sin\theta) + i\sin(x\cos\theta). \end{aligned}$$

Substituting this into Eq. (7.87) we obtain

$$\begin{aligned} \cos(x\sin\theta) + i\sin(x\cos\theta) &= \sum_{n=-\infty}^{\infty} J_n(x)(\cos\theta + i\sin\theta)^n \\ &= \sum_{-\infty}^{\infty} J_n(x)\cos n\theta + i\sum_{-\infty}^{\infty} J_n(x)\sin n\theta. \end{aligned}$$

Since $J_{-n}(x) = (-1)^n J_n(x), \cos n\theta = \cos(-n\theta)$, and $\sin n\theta = -\sin(-n\theta)$, we have, upon equating the real and imaginary parts of the above equation,

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta,$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin(2n-1)\theta.$$

It is interesting to note that these are the Fourier cosine and sine series of $\cos(x\sin\theta)$ and $\sin(x\sin\theta)$. Multiplying the first equation by $\cos k\theta$ and integrating from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos k\theta \cos(x\sin\theta)d\theta = \begin{cases} J_k(x), & \text{if } k = 0, 2, 4, \ldots \\ 0, & \text{if } k = 1, 3, 5, \ldots \end{cases}.$$

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Now multiplying the second equation by $\sin k\theta$ and integrating from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^\pi \sin k\theta \sin(x\sin\theta)d\theta = \begin{cases} J_k(x), & \text{if } k = 1, 3, 5, \ldots \\ 0, & \text{if } k = 0, 2, 4, \ldots \end{cases}.$$

Adding these two together we obtain Bessel's integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)d\theta, \quad n = \text{positive integer}. \tag{7.89}$$

Recurrence formulas for $J_n(x)$

Bessel functions of the first kind, $J_n(x)$, are the most useful, because they are bounded near the origin. And there exist some useful recurrence formulas between Bessel functions of different orders and their derivatives.

(1) $$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x). \tag{7.90}$$

Proof: Differentiating both sides of the generating function with respect to t, we obtain

$$e^{x(t-t^{-1})/2}\frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}$$

or

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}.$$

This can be rewritten as

$$\frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n-2} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}$$

or

$$\frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_{n+2}(x)t^n = \sum_{n=-\infty}^{\infty} (n+1)J_{n+1}(x)t^n.$$

Equating coefficients of t^n on both sides, we obtain

$$\frac{x}{2}J_n(x) + \frac{x}{2}J_{n+2}(x) = (n+1)J_n(x).$$

Replacing n by $n-1$, we obtain the required result.

(2) $$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x). \tag{7.91}$$

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Proof:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)2^{n+2k}} x^{n+2k}.$$

Differentiating both sides once, we obtain

$$J_n'(x) = \sum_{k=0}^{\infty} \frac{(n+2k)(-1)^k}{k!\Gamma(n+k+1)2^{n+2k}} x^{n+2k-1},$$

from which we have

$$xJ_n'(x) = nJ_n(x) + x\sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!\Gamma(n+k+1)2^{n+2k-1}} x^{n+2k-1}.$$

Letting $k = m+1$ in the sum on the right hand side, we obtain

$$\begin{aligned} xJ_n'(x) &= nJ_n(x) - x\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+2)2^{n+2m+1}} x^{n+2m+1} \\ &= nJ_n(x) - xJ_{n+1}(x). \end{aligned}$$

(3) $\quad xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x).$ (7.92)

Proof: Differentiating both sides of the following equation with respect to x

$$x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)2^{n+2k}} x^{2n+2k},$$

we have

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_n'(x) + nx^{n-1}J_n(x),$$

$$\begin{aligned} \frac{d}{dx}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2n+2k}}{2^{n+2k}k!\Gamma(n+k+1)} &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2n+2k-1}}{2^{n+2k-1}k!\Gamma(n+k)} \\ &= x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{(n-1)+2k}}{2^{(n-1)+2k}k!\Gamma[(n-1)+k+1]} \\ &= x^n J_{n-1}(x). \end{aligned}$$

Equating these two results, we have

$$x^n J_n'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x).$$

Canceling out the common factor x^{n-1}, we obtained the required result (7.92).

(4) $J_n'(x) = [J_{n-1}(x) - J_{n+1}(x)]/2.$ (7.93)

Proof: Adding (7.91) and (7.92) and dividing by $2x$, we obtain the required result (7.93).

If we subtract (7.91) from (7.92), $J_n'(x)$ is eliminated and we obtain

$$xJ_{n+1}(x) + xJ_{n-1}(x) = 2nJ_n(x)$$

which is Eq. (7.90).

These recurrence formulas (or important identities) are very useful. Here are some illustrative examples.

Example 7.2

Show that $J_0'(x) = J_{-1}(x) = -J_1(x)$.

Solution: From Eq. (7.93), we have

$$J_0'(x) = [J_{-1}(x) - J_1(x)]/2,$$

then using the fact that $J_{-n}(x) = (-1)^n J_n(x)$, we obtain the required results.

Example 7.3

Show that

$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x).$$

Solution: Letting $n = 4$ in (7.90), we have

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x).$$

Similarly, for $J_2(x)$ we have

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x).$$

Substituting this into the expression for $J_3(x)$, we obtain the required result.

Example 7.4

Find $\int_0^t xJ_0(x)dx$.

Solution: Taking derivative of the quantity $xJ_1(x)$ with respect to x, we obtain

$$\frac{d}{dx}\{xJ_1(x)\} = J_1(x) + xJ_1'(x).$$

Then using Eq. (7.92) with $n = 1$, $xJ_1'(x) = -J_1(x) + xJ_0(x)$, we find

$$\frac{d}{dx}\{xJ_1(x)\} = J_1(x) + xJ_1'(x) = xJ_0(x),$$

thus,

$$\int_0^t xJ_0(x)dx = xJ_1(x)|_0^t = tJ_1(t).$$

Approximations to the Bessel functions

For very large or very small values of x we might be able to make some approximations to the Bessel functions of the first kind $J_n(x)$. By a rough argument, we can see that the Bessel functions behave something like a damped cosine function when the value of x is very large. To see this, let us go back to Bessel's equation (7.71)

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

and rewrite it as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0.$$

If x is very large, let us drop the term α^2/x^2 and then the differential equation reduces to

$$y'' + \frac{1}{x}y' + y = 0.$$

Let $u = yx^{1/2}$, then $u' = y'x^{1/2} + \frac{1}{2}x^{-1/2}y$, and $u'' = y''x^{1/2} + x^{-1/2}y' - \frac{1}{4}x^{-3/2}y$. From u'' we have

$$y'' + \frac{1}{x}y' = x^{-1/2}u'' + \frac{1}{4x^2}y.$$

Adding y on both sides, we obtain

$$y'' + \frac{1}{x}y' + y = 0 = x^{-1/2}u'' + \frac{1}{4x^2}y + y,$$

$$x^{-1/2}u'' + \frac{1}{4x^2}y + y = 0$$

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or

$$u'' + \left(\frac{1}{4x^2} + 1\right)x^{1/2}y = u'' + \left(\frac{1}{4x^2} + 1\right)u = 0,$$

the solution of which is

$$u = A\cos x + B\sin x.$$

Thus the approximate solution to Bessel's equation for very large values of x is

$$y = x^{-1/2}(A\cos x + B\sin x) = Cx^{-1/2}\cos(x + \beta).$$

A more rigorous argument leads to the following asymptotic formula

$$J_n(x) \approx \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right). \tag{7.94}$$

For very small values of x (that is, near 0), by examining the solution itself and dropping all terms after the first, we find

$$J_n(x) \approx \frac{x^n}{2^n\Gamma(n+1)}. \tag{7.95}$$

Orthogonality of Bessel functions

Bessel functions enjoy a property which is called orthogonality and is of general importance in mathematical physics. If λ and μ are two different constants, we can show that under certain conditions

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx = 0.$$

Let us see what these conditions are. First, we can show that

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx = \frac{\mu J_n(\lambda)J_n'(\mu) - \lambda J_n(\mu)J_n'(\lambda)}{\lambda^2 - \mu^2}. \tag{7.96}$$

To show this, let us go back to Bessel's equation (7.71) and change the independent variable to λx, where λ is a constant, then the resulting equation is

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0$$

and its general solution is $J_n(\lambda x)$. Now suppose we have two such equations, one for y_1 with constant λ, and one for y_2 with constant μ:

$$x^2y_1'' + xy_1' + (\lambda^2x^2 - n^2)y_1 = 0, \quad x^2y_2'' + xy_2' + (\mu^2x^2 - n^2)y_2 = 0.$$

Now multiplying the first equation by y_2, the second by y_1 and subtracting, we get

$$x^2[y_2y_1'' - y_1y_2''] + x[y_2y_1' - y_1y_2'] = (\mu^2 - \lambda^2)x^2y_1y_2.$$

Dividing by x we obtain

$$x\frac{d}{dx}[y_2y_1' - y_1y_2'] + [y_2y_1' - y_1y_2'] = (\mu^2 - \lambda^2)xy_1y_2$$

or

$$\frac{d}{dx}\{x[y_2y_1' - y_1y_2']\} = (\mu^2 - \lambda^2)xy_1y_2$$

and then integration gives

$$(\mu^2 - \lambda^2)\int xy_1y_2dx = x[y_2y_1' - y_1y_2'],$$

where we have omitted the constant of integration. Now $y_1 = J_n(\lambda x), y_2 = J_n(x)$, and if $\lambda \neq \mu$ we then have

$$\int xJ_n(\lambda x)J_n(\mu x)dx = \frac{x[\lambda J_n(\mu x)J_n'(\lambda x) - \mu J_n(\lambda x)J_n'(\mu x)}{\mu^2 - \lambda^2}.$$

Thus

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx = \frac{\mu J_n(\lambda)J_n'(\mu) - \lambda J_n(\mu)J_n'(\lambda)}{\lambda^2 - \mu^2} \quad \text{q.e.d.}$$

Now letting $\mu \to \lambda$ and using L'Hospital's rule, we obtain

$$\int_0^1 xJ_n^2(\lambda x)dx = \lim_{\mu\to\lambda}\frac{\lambda J_n'(\mu)J_n'(\lambda) - J_n(\lambda)J_n'(\mu) - \mu J_n(\lambda)J_n''(\mu)}{2\mu}$$
$$= \frac{\lambda J_n'^2(\lambda) - J_n(\lambda)J_n'(\lambda) - \lambda J_n(\lambda)J_n''(\lambda)}{2\lambda}.$$

But

$$\lambda^2 J_n''(\lambda) + \lambda J_n'(\lambda) + (\lambda^2 - n^2)J_n(\lambda) = 0.$$

Solving for $J_n''(\lambda)$ and substituting, we obtain

$$\int_0^1 xJ_n^2(\lambda x)dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^2(x)\right]. \tag{7.97}$$

Furthermore, if λ and μ are any two different roots of the equation $RJ_n(x) + SxJ_n'(x) = 0$, where R and S are constant, we then have

$$RJ_n(\lambda) + S\lambda J_n'(\lambda) = 0, \quad RJ_n(\mu) + S\mu J_n'(\mu) = 0;$$

from these two equations we find, if $R \neq 0, S \neq 0$,

$$\mu J_n(\lambda)J_n'(\mu) - \lambda J_n(\mu)J_n'(\lambda) = 0$$

and then from Eq. (7.96) we obtain

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx = 0. \tag{7.98}$$

Thus, the two functions $\sqrt{x}J_n(\lambda x)$ and $\sqrt{x}J_n(\mu x)$ are orthogonal in (0, 1). We can also say that the two functions $J_n(\lambda x)$ and $J_n(\mu x)$ are orthogonal with respect to the weighted function x.

Eq. (7.98) is also easily proved if $R = 0$ and $S \neq 0$, or $R \neq 0$ but $S = 0$. In this case, λ and μ can be any two different roots of $J_n(x) = 0$ or $J_n'(x) = 0$.

Spherical Bessel functions

In physics we often meet the following equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + [k^2r^2 - l(l+1)]R = 0, \qquad (l = 0, 1, 2, \ldots). \tag{7.99}$$

In fact, this is the radial equation of the wave and the Helmholtz partial differential equation in the spherical coordinate system (see Problem 7.22). If we let $x = kr$ and $y(x) = R(r)$, then Eq. (7.99) becomes

$$x^2y'' + 2xy' + [x^2 - l(l+1)]y = 0 \qquad (l = 0, 1, 2, \ldots), \tag{7.100}$$

where $y' = dy/dx$. This equation almost matches Bessel's equation (7.71). Let us make the further substitution

$$y(x) = w(x)/\sqrt{x},$$

then we obtain

$$x^2w'' + xw' + [x^2 - (l + \tfrac{1}{2})]w = 0 \qquad (l = 0, 1, 2, \ldots). \tag{7.101}$$

The reader should recognize this equation as Bessel's equation of order $l + \frac{1}{2}$. It follows that the solutions of Eq. (7.100) can be written in the form

$$y(x) = A\frac{J_{l+1/2}(x)}{\sqrt{x}} + B\frac{J_{-l-1/2}(x)}{\sqrt{x}}.$$

This leads us to define spherical Bessel functions $j_l(x) = CJ_{l+E}(x)/\sqrt{x}$. The factor C is usually chosen to be $\sqrt{\pi/2}$ for a reason to be explained later:

$$j_l(x) = \sqrt{\pi/2x}\,J_l + E(x). \tag{7.102}$$

Similarly, we can define

$$n_l(x) = \sqrt{\pi/2x}\,N_{l+E}(x).$$

We can express $j_l(x)$ in terms of $j_0(x)$. To do this, let us go back to $J_n(x)$ and we find that

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x), \quad \text{or} \quad J_{n+1}(x) = -x^n\frac{d}{dx}\{x^{-n}J_n(x)\}.$$

The proof is simple and straightforward:

$$\begin{aligned}\frac{d}{dx}\{x^{-n}J_n(x)\} &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{n+2k}k!\Gamma(n+k+1)} \\ &= x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k x^{n+2k-1}}{2^{n+2k-1}(k-1)!\Gamma(n+k+1)} \\ &= x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^{k+1} x^{n+2k+1}}{2^{n+2k+1}k!\Gamma[(n+k+2\}} = -x^{-n}J_{n+1}(x).\end{aligned}$$

Now if we set $n = l + \frac{1}{2}$ and divide by $x^{l+3/2}$, we obtain

$$\frac{J_{l+3/2}(x)}{x^{l+3/2}} = -\frac{1}{x}\frac{d}{dx}\left[\frac{J_{l+1/2}(x)}{x^{l+1/2}}\right] \quad \text{or} \quad \frac{j_{l+1}(x)}{x^{l+1}} = -\frac{1}{x}\frac{d}{dx}\left[\frac{j_l(x)}{x^l}\right].$$

Starting with $l = 0$ and applying this formula l times, we obtain

$$j_l(x) = x^l\left(-\frac{1}{x}\frac{d}{dx}\right)^l j_0(x) \qquad (l = 1, 2, 3, \ldots). \tag{7.103}$$

Once $j_0(x)$ has been chosen, all $j_l(x)$ are uniquely determined by Eq. (7.103).

Now let us go back to Eq. (7.102) and see why we chose the constant factor C to be $\sqrt{\pi/2}$. If we set $l = 0$ in Eq. (7.101), the resulting equation is

$$xy'' + 2y' + xy = 0.$$

Solving this equation by the power series method, the reader will find that functions $\sin(x)/x$ and $\cos(x)/x$ are among the solutions. It is customary to define

$$j_0(x) = \sin(x)/x.$$

Now by using Eq. (7.76), we find

$$\begin{aligned}J_{1/2}(x) &= \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{1/2+2k}}{k!\Gamma(k+3/2)} \\ &= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x.\end{aligned}$$

Comparing this with $j_0(x)$ shows that $j_0(x) = \sqrt{\pi/2x}J_{1/2}(x)$, and this explains the factor $\sqrt{\pi/2}$ chosen earlier.

Sturm–Liouville systems

A boundary-value problem having the form

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [q(x) + \lambda p(x)]y = 0, \qquad a \le x \le b \tag{7.104}$$

and satisfying boundary conditions of the form

$$k_1 y(a) + k_2 y'(a) = 0, \qquad l_1 y(b) + l_2 y'(b) = 0 \tag{7.104a}$$

is called a Sturm–Liouville boundary-value problem; Eq. (7.104) is known as the Sturm–Liouville equation. Legendre's equation, Bessel's equation and many other important equations can be written in the form of (7.104).

Legendre's equation (7.1) can be written as

$$[(1 - x^2)y']' + \lambda y = 0, \qquad \lambda = \nu(\nu + 1);$$

we can then see it is a Sturm–Liouville equation with $r = 1 - x^2, q = 0$ and $p = 1$.

Then, how do Bessel functions fit into the Sturm–Liouville framework? $J(s)$ satisfies the Bessel equation (7.71)

$$s^2\ddot{J}_n + s\dot{J}_n + (s^2 - n^2)J_n = 0, \qquad \dot{J}_n = dJ_n/ds. \tag{7.71a}$$

We assume n is a positive integer and setting $s = \lambda x$, with λ a non-zero constant, we have

$$\frac{ds}{dx} = \lambda, \quad \dot{J}_n = \frac{dJ_n}{dx}\frac{dx}{ds} = \frac{1}{\lambda}\frac{dJ_n}{dx}, \quad \ddot{J}_n = \frac{d}{dx}\left(\frac{1}{\lambda}\frac{dJ_n}{dx}\right)\frac{dx}{ds} = \frac{1}{\lambda^2}\frac{d^2J_n}{dx^2}$$

and Eq. (7.71a) becomes

$$x^2 J_n''(\lambda x) + xJ_n'(\lambda x) + (\lambda^2 x^2 - n^2)J_n(\lambda x) = 0, \quad J_n' = dJ_n/dx$$

or

$$xJ_n''(\lambda x) + J_n'(\lambda x) + (\lambda^2 x - n^2/x)J_n(\lambda x) = 0,$$

which can be written as

$$[xJ_n'(\lambda x)]' + \left(-\frac{n^2}{x} + \lambda^2 x\right)J_n(\lambda x) = 0.$$

It is easy to see that for each fixed n this is a Sturm–Liouville equation (7.104), with $r(x) = x$, $q(x) = -n^2/x, p(x) = x$, and with the parameter λ now written as λ^2.

For the Sturm–Liouville system (7.104) and (7.104a), a non-trivial solution exists in general only for a particular set of values of the parameter λ. These values are called the eigenvalues of the system. If $r(x)$ and $q(x)$ are real, the eigenvalues are real. The corresponding solutions are called eigenfunctions of the system. In general there is one eigenfunction to each eigenvalue. This is the non-degenerate case. In the degenerate case, more than one eigenfunction may correspond to the same eigenvalue. The eigenfunctions form an orthogonal set with respect to the density function $p(x)$ which is generally ≥ 0.Thus by suitable normalization the set of functions can be made an orthonormal set with respect to $p(x)$ in $a \leq x \leq b$. We now proceed to prove these two general claims.

Property 1
If $r(x)$ and $q(x)$ are real, the eigenvalues of a Sturm–Liouville system are real.

We start with the Sturm–Liouville equation (7.104) and the boundary conditions (7.104a):

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [q(x) + \lambda p(x)]y = 0, \qquad a \leq x \leq b,$$

$$k_1 y(a) + k_2 y'(a) = 0, \quad l_1 y(b) + l_2 y'(b) = 0,$$

and assume that $r(x), q(x), p(x), k_1, k_2, l_1$, and l_2 are all real, but λ and y may be complex. Now take the complex conjugates

$$\frac{d}{dx}\left[r(x)\frac{d\bar{y}}{dx}\right] + [q(x) + \bar{\lambda} p(x)]\bar{y} = 0, \tag{7.105}$$

$$k_1 \bar{y}(a) + k_2 \bar{y}'(a) = 0, l_1 \bar{y}(b) + l_2 \bar{y}'(b) = 0, \tag{7.105a}$$

where $\bar{y}$ and $\bar{\lambda}$ are the complex conjugates of y and λ, respectively.

Multiplying (7.104) by $\bar{y}$, (7.105) by y, and subtracting, we obtain after simplifying

$$\frac{d}{dx}\left[r(x)(y\bar{y}' - \bar{y}y')\right] = (\lambda - \bar{\lambda})p(x)y\bar{y}.$$

Integrating from a to b, and using the boundary conditions (7.104a) and (7.105a), we then obtain

$$(\lambda - \bar{\lambda})\int_a^b p(x)\left|y'\right|^2 dx = r(x)(y\bar{y}' - \bar{y}y')\Big|_a^b = 0.$$

Since $p(x) \geq 0$ in $a \leq x \leq b$, the integral on the left is positive and therefore $\lambda = \bar{\lambda}$, that is, λ is real.

Property 2
The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to $p(x)$ in $a \leq x \leq b$.

If y_1 and y_2 are eigenfunctions corresponding to the two different eigenvalues λ_1, λ_2, respectively,

$$\frac{d}{dx}\left[r(x)\frac{dy_1}{dx}\right] + [q(x) + \lambda_1 p(x)]y_1 = 0, \qquad a \le x \le b, \tag{7.106}$$

$$k_1 y_1(a) + k_2 y_1'(a) = 0, \quad l_1 y_1(b) + l_2 y_1'(b) = 0; \tag{7.106a}$$

$$\frac{d}{dx}\left[r(x)\frac{dy_2}{dx}\right] + [q(x) + \lambda_2 p(x)]y_2 = 0, \qquad a \le x \le b, \tag{7.107}$$

$$k_1 y_2(a) + k_2 y_2'(a) = 0, \qquad l_1 y_2(b) + l_2 y_2'(b) = 0. \tag{7.107a}$$

Multiplying (7.106) by y_2 and (7.107) by y_1, then subtracting, we obtain

$$\frac{d}{dx}\left[r(x)(y_1 y_2' - y_2 y_1')\right] = (\lambda - \bar{\lambda})p(x)y_1 y_2.$$

Integrating from a to b, and using (7.106a) and (7.107a), we obtain

$$(\lambda_1 - \lambda_2)\int_a^b p(x)y_1 y_2 dx = r(x)(y_1 y_2' - y_2 y_1')|_a^b = 0.$$

Since $\lambda_1 \ne \lambda_2$ we have the required result; that is,

$$\int_a^b p(x)y_1 y_2 dx = 0.$$

We can normalize these eigenfunctions to make them an orthonormal set, and so we can expand a given function in a series of these orthonormal eigenfunctions.

We have shown that Legendre's equation is a Sturm–Liouville equation with $r(x) = 1 - x, q = 0$ and $p = 1$. Since $r = 0$ when $x = \pm 1$, no boundary conditions are needed to form a Sturm–Liouville problem on the interval $-1 \le x \le 1$. The numbers $\lambda_n = n(n+1)$ are eigenvalues with $n = 0, 1, 2, 3, \ldots$. The corresponding eigenfunctions are $y_n = P_n(x)$. Property 2 tells us that

$$\int_{-1}^1 P_n(x)P_m(x)dx = 0 \qquad n \ne m.$$

For Bessel functions we saw that

$$[xJ_n'(\lambda x)]' + \left(-\frac{n^2}{x} + \lambda^2 x\right)J_n(\lambda x) = 0$$

is a Sturm–Liouville equation (7.104), with $r(x) = x, q(x) = -n^2/x, p(x) = x$, and with the parameter λ now written as λ^2. Typically, we want to solve this equation on an interval $0 \le x \le b$ subject to

$$J_n(\lambda b) = 0.$$

which limits the selection of λ. Property 2 then tells us that

$$\int_0^b x J_n(\lambda_k x) J_n(\lambda_l x) dx = 0, \qquad k \ne l.$$

Problems

7.1 Using Eq. (7.11), show that $P_n(-x) = (-1)^n P_n(x)$ and $P_n{}'(-x) = (-1)^{n+1} P_n'(x)$.

7.2 Find $P_0(x), P_1(x), P_2(x), P_3(x)$, and $P_4(x)$ from Rodrigues' formula (7.12). Compare your results with Eq. (7.11).

7.3 Establish the recurrence formula (7.16b) by manipulating Rodrigues' formula.

7.4 Prove that $P_5'(x) = 9P_4(x) + 5P_2(x) + P_0(x)$.
Hint: Use the recurrence relation (7.16d).

7.5 Let P and Q be two points in space (Fig. 7.6). Using Eq. (7.14), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}}$$
$$= \frac{1}{r_2}\left[P_0 + P_1(\cos\theta)\frac{r_1}{r_2} + P_2(\cos\theta)\left(\frac{r_1}{r_2}\right)^2 + \cdots\right].$$

7.6 What is $P_n(1)$? What is $P_n(-1)$?

7.7 Obtain the associated Legendre functions: (*a*) $P_2^1(x)$, (*b*) $P_3^2(x)$, (*c*) $P_2^3(x)$.

7.8 Verify that $P_3^2(x)$ is a solution of Legendre's associated equation (7.25) for $m = 2$, $n = 3$.

7.9 Verify the orthogonality conditions (7.31) for the functions $P_2^1(x)$ and $P_3^1(x)$.

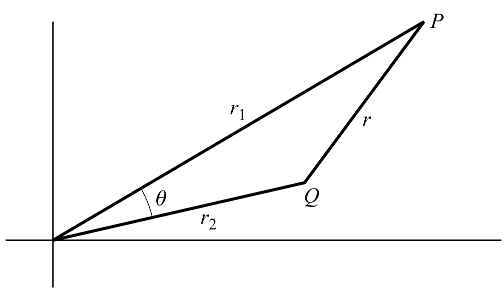

Figure 7.6.

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7.10 Verify Eq. (7.37) for the function $P_2^1(x)$.

7.11 Show that

$$\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n = \frac{(n-m)!}{(n+m)!}(x^2-1)^m \frac{d^{n+m}}{dx^{n+m}}(x^2-1)^m$$

Hint: Write $(x^2-1)^n = (x-1)^n(x+1)^n$ and find the derivatives by Leibnitz's rule.

7.12 Use the generating function for the Hermite polynomials to find: (a) $H_0(x)$; (b) $H_1(x)$; (c) $H_2(x)$; (d) $H_3(x)$.

7.13 Verify that the generating function Φ satisfies the identity

$$\frac{\partial^2\Phi}{\partial x^2} - 2x\frac{\partial\Phi}{\partial x} + 2t\frac{\partial\Phi}{\partial t} = 0.$$

Show that the functions $H_n(x)$ in Eq. (7.47) satisfy Eq. (7.38).

7.14 Given the differential equation $y'' + (\varepsilon - x^2)y = 0$, find the possible values of ε (eigenvalues) such that the solution $y(x)$ of the given differential equation tends to zero as $x \to \pm\infty$. For these values of ε, find the eigenfunctions $y(x)$.

7.15 In Eq. (7.58), write the series for the exponential and collect powers of z to verify the first few terms of the series. Verify the identity

$$x\frac{\partial^2\Phi}{\partial x^2} + (1-x)\frac{\partial\Phi}{\partial x} + z\frac{\partial\Phi}{\partial z} = 0.$$

Substituting the series (7.58) into this identity, show that the functions $L_n(x)$ in Eq. (7.58) satisfy Laguerre's equation.

7.16 Show that

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots,$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + - \cdots.$$

7.17 Show that

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin x, J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos x.$$

7.18 If n is a positive integer, show that the formal expression for $J_{-n}(x)$ gives $J_{-n}(x) = (-1)^n J_n(x)$.

7.19 Find the general solution to the modified Bessel's equation

$$x^2y'' + xy' + (x^2s^2 - \alpha^2)y = 0$$

which differs from Bessel's equation only in that sx takes the place of x.

(Hint: Reduce the given equation to Bessel's equation first.)

7.20 The lengthening simple pendulum: Consider a small mass m suspended by a string of length l. If its length is increased at a steady rate r as it swings back and forth freely in a vertical plane, find the equation of motion and the solution for small oscillations.

7.21 Evaluate the integrals:

$$(a) \int x^n J_{n-1}(x)dx; \qquad (b) \int x^{-n} J_{n+1}(x)dx; \qquad (c) \int x^{-1} J_1(x)dx.$$

7.22 In quantum mechanics, the three-dimensional Schrödinger equation is

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(\mathbf{r}, t) + V\psi(\mathbf{r}, t), \qquad i = \sqrt{-1}, \hbar = h/2\pi.$$

(a) When the potential V is independent of time, we can write $\psi(\mathbf{r}, t) = u(\mathbf{r})T(t)$. Show that in this case the Schrödinger equation reduces to

$$-\frac{\hbar^2}{2m}\nabla^2 u(\mathbf{r}) + Vu(\mathbf{r}) = Eu(\mathbf{r}),$$

a time-independent equation along with $T(t) = e^{-iEt/\hbar}$, where E is a separation constant.

(b) Show that, in spherical coordinates, the time-independent Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi}\right] + V(r)u = Eu,$$

then use separation of variables, $u(r, \theta, \phi) = R(r)Y(\theta, \phi)$, to split it into two equations, with α as a new separation constant:

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(V + \frac{\alpha}{r^2}\right)R = ER,$$

$$-\frac{\hbar^2}{2m}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) - \frac{\hbar^2}{2m}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = \alpha Y.$$

It is straightforward to see that the radial equation is in the form of Eq. (7.99). Continuing the separation process by putting $Y(\theta, \phi) = \Theta(\theta)\Phi(\theta)$, the angular equation can be separated further into two equations, with β as separation constant:

$$-\frac{\hbar^2}{2m}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \beta,$$

$$-\frac{\hbar^2}{2m}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \alpha\sin^2\theta\Theta + \beta\Theta = 0.$$

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The first equation is ready for integration. Do you recognize the second equation in θ as Legendre's equation? (Compare it with Eq. (7.30).) If you are unsure, try to simplify it by putting $\gamma = 2m\alpha/\hbar$, $\mu = (2m\beta/\hbar)^{1/2}$, and you will obtain

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + (\gamma \sin^2\theta - \mu^2)\Theta = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left(\gamma - \frac{\mu^2}{\sin^2\theta}\right)\Theta = 0,$$

which more closely resembles Eq. (7.30).

7.23 Consider the differential equation

$$y'' + R(x)y' + [Q(x) + \lambda P(x)]y = 0.$$

Show that it can be put into the form of the Sturm–Liouville equation (7.104) with

$$r(x) = e^{\int R(x)dx}, \quad q(x) = Q(x)e^{\int R(x)dx}, \quad \text{and} \quad p(x) = P(x)e^{\int R(x)dx}.$$

7.24. (*a*) Show that the system $y'' + \lambda y = 0, y(0) = 0, y(1) = 0$ is a Sturm–Liouville system.

(*b*) Find the eigenvalues and eigenfunctions of the system.

(*c*) Prove that the eigenfunctions are orthogonal on the interval $0 \le x \le 1$.

(*d*) Find the corresponding set of normalized eigenfunctions, and expand the function $f(x) = 1$ in a series of these orthonormal functions.