9

The Laplace transformation

The Laplace transformation method is generally useful for obtaining solutions of linear differential equations (both ordinary and partial). It enables us to reduce a differential equation to an algebraic equation, thus avoiding going to the trouble of finding the general solution and then evaluating the arbitrary constants. This procedure or technique can be extended to systems of equations and to integral equations, and it often yields results more readily than other techniques. In this chapter we shall first define the Laplace transformation, then evaluate the transformation for some elementary functions, and finally apply it to solve some simple physical problems.

Definition of the Lapace transform

The Laplace transform $L[f(x)]$ of a function $f(x)$ is defined by the integral

$$L[f(x)] = \int_0^\infty e^{-px} f(x)dx = F(p), \tag{9.1}$$

whenever this integral exists. The integral in Eq. (9.1) is a function of the parameter p and we denote it by $F(p)$. The function $F(p)$ is called the Laplace transform of $f(x)$. We may also look upon Eq. (9.1) as a definition of a Laplace transform operator L which tranforms $f(x)$ in to $F(p)$. The operator L is linear, since from Eq. (9.1) we have

$$\begin{aligned} L[c_1 f(x) + c_2 g(x)] &= \int_0^\infty e^{-px}\{c_1 f(x) + c_2 g(x)\}dx \\ &= c_1 \int_0^\infty e^{-px} f(x)dx + c_2 \int_0^\infty e^{-px} g(x)dx \\ &= c_1 L[f(x)] + c_2 L[g(x)], \end{aligned}$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

where c_1 and c_2 are arbitrary constants and $g(x)$ is an arbitrary function defined for $x > 0$.

The inverse Laplace transform of $F(p)$ is a function $f(x)$ such that $L[f(x)] = F(p)$. We denote the operation of taking an inverse Laplace transform by L^{-1}:

$$L^{-1}[F(p)] = f(x). \tag{9.2}$$

That is, we operate algebraically with the operators L and L^{-1}, bringing them from one side of an equation to the other side just as we would in writing $ax = b$ implies $x = a^{-1}b$. To illustrate the calculation of a Laplace transform, let us consider the following simple example.

Example 9.1
Find $L[e^{ax}]$, where a is a constant.

Solution: The transform is

$$L[e^{ax}] = \int_0^\infty e^{-px}e^{ax}dx = \int_0^\infty e^{-(p-a)x}dx.$$

For $p \le a$, the exponent on e is positive or zero and the integral diverges. For $p > a$, the integral converges:

$$L[e^{ax}] = \int_0^\infty e^{-px}e^{ax}dx = \int_0^\infty e^{-(p-a)x}dx = \left.\frac{e^{-(p-a)x}}{-(p-a)}\right|_0^\infty = \frac{1}{p-a}.$$

This example enables us to investigate the existence of Eq. (9.1) for a general function $f(x)$.

Existence of Laplace transforms

We can prove that:

(1) if $f(x)$ is piecewise continuous on every finite interval $0 \le x \le X$, and
(2) if we can find constants M and a such that $|f(x)| \le Me^{ax}$ for $x \ge X$,

then $L[f(x)]$ exists for $p > a$. A function $f(x)$ which satisfies condition (2) is said to be of *exponential order* as $x \to \infty$; this is mathematician's jargon!

These are sufficient conditions on $f(x)$ under which we can guarantee the existence of $L[f(x)]$. Under these conditions the integral converges for $p > a$:

$$\left|\int_0^X f(x)e^{-px}dx\right| \le \int_0^X |f(x)|e^{-px}dx \le \int_0^X Me^{ax}e^{-px}dx$$
$$\le M\int_0^\infty e^{-(p-a)x}dx = \frac{M}{p-a}.$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

This establishes not only the convergence but the absolute convergence of the integral defining $L[f(x)]$. Note that $M/(p-a)$ tends to zero as $p \to \infty$. This shows that

$$\lim_{p\to\infty} F(p) = 0 \tag{9.3}$$

for all functions $F(p) = L[f(x)]$ such that $f(x)$ satisfies the foregoing conditions (1) and (2). It follows that if $\lim_{p\to\infty} F(p) \neq 0$, $F(p)$ cannot be the Laplace transform of any function $f(x)$.

It is obvious that functions of exponential order play a dominant role in the use of Laplace transforms. One simple way of determining whether or not a specified function is of exponential order is the following one: if a constant b exists such that

$$\lim_{x\to\infty} \left[e^{-bx}|f(x)|\right] \tag{9.4}$$

exists, the function $f(x)$ is of exponential order (of the order of e^{-bx}). To see this, let the value of the above limit be $K \neq 0$. Then, when x is large enough, $|e^{-bx}f(x)|$ can be made as close to K as possible, so certainly

$$|e^{-bx}f(x)| < 2K.$$

Thus, for sufficiently large x,

$$|f(x)| < 2Ke^{bx}$$

or

$$|f(x)| < Me^{bx}, \quad \text{with} \quad M = 2K.$$

On the other hand, if

$$\lim_{x\to\infty} [e^{-cx}|f(x)|] = \infty \tag{9.5}$$

for every fixed c, the function $f(x)$ is not of exponential order. To see this, let us assume that b exists such that

$$|f(x)| < Me^{bx} \quad \text{for} \quad x \geq X$$

from which it follows that

$$|e^{-2bx}f(x)| < Me^{-bx}.$$

Then the choice of $c = 2b$ would give us $|e^{-cx}f(x)| < Me^{-bx}$, and $e^{-cx}f(x) \to 0$ as $x \to \infty$ which contradicts Eq. (9.5).

Example 9.2
Show that x^3 is of exponential order as $x \to \infty$.

Solution: We have to check whether or not

$$\lim_{x\to\infty}\left(e^{-bx}x^3\right) = \lim_{x\to\infty}\frac{x^3}{e^{bx}}$$

exists. Now if $b > 0$, then L'Hospital's rule gives

$$\lim_{x\to\infty}\left(e^{-bx}x^3\right) = \lim_{x\to\infty}\frac{x^3}{e^{bx}} = \lim_{x\to\infty}\frac{3x^2}{be^{bx}} = \lim_{x\to\infty}\frac{6x}{b^2e^{bx}} = \lim_{x\to\infty}\frac{6}{b^3e^{bx}} = 0.$$

Therefore x^3 is of exponential order as $x \to \infty$.

Laplace transforms of some elementary functions

Using the definition (9.1) we now obtain the transforms of polynomials, exponential and trigonometric functions.

(1) $f(x) = 1$ for $x > 0$.
By definition, we have

$$L[1] = \int_0^\infty e^{-px}dx = \frac{1}{p}, \qquad p > 0.$$

(2) $f(x) = x^n$, where n is a positive integer.

By definition, we have

$$L[x^n] = \int_0^\infty e^{-px}x^n dx.$$

Using integration by parts:

$$\int uv'dx = uv - \int vu'dx$$

with

$$u = x^n, \quad dv = v'dx = e^{-px}dx = -(1/p)d(e^{-px}), \quad v = -(1/p)e^{-px},$$

we obtain

$$\int_0^\infty e^{-px}x^n dx = \left[\frac{-x^n e^{-px}}{p}\right]_0^\infty + \frac{n}{p}\int_0^\infty e^{-px}x^{n-1}dx.$$

For $p > 0$ and $n > 0$, the first term on the right hand side of the above equation is zero, and so we have

$$\int_0^\infty e^{-px}x^n dx = \frac{n}{p}\int_0^\infty e^{-px}x^{n-1}dx$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

or

$$L[x^n] = \frac{n}{p} L[x^{n-1}]$$

from which we may obtain for $n > 1$

$$L[x^{n-1}] = \frac{n-1}{p} L[x^{n-2}].$$

Iteration of this process yields

$$L[x^n] = \frac{n(n-1)(n-2)\cdots 2 \cdot 1}{p^n} L[x^0].$$

By (1) above we have

$$L[x^0] = L[1] = 1/p.$$

Hence we finally have

$$L[x^n] = \frac{n!}{p^{n+1}}, \qquad p > 0.$$

(3) $f(x) = e^{ax}$, where a is a real constant.

$$L[e^{ax}] = \int_0^\infty e^{-px} e^{ax} dx = \frac{1}{p-a},$$

where $p > a$ for convegence. (For details, see Example 9.1.)

(4) $f(x) = \sin ax$, where a is a real constant.

$$L[\sin ax] = \int_0^\infty e^{-px} \sin ax dx.$$

Using

$$\int uv' dx = uv - \int vu' dx \quad \text{with} \qquad u = e^{-px}, \quad dv = -d(\cos ax)/a,$$

and

$$\int e^{mx} \sin nx dx = \frac{e^{mx}(m \sin nx - n \cos nx)}{n^2 + m^2}$$

(you can obtain this simply by using integration by parts twice) we obtain

$$L[\sin ax] = \int_0^\infty e^{-px} \sin ax dx = \left[\frac{e^{-px}(-p \sin ax - a \cos ax)}{p^2 + a^2} \right]_0^\infty.$$

Since p is positive, $e^{-px} \to 0$ as $x \to \infty$, but sin ax and cos ax are bounded as $x \to \infty$, so we obtain

$$L[\sin ax] = 0 - \frac{1(0-a)}{p^2 + a^2} = \frac{a}{p^2 + a^2}, \quad p > 0.$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

(5) $f(x) = \cos ax$, where a is a real constant.

Using the result

$$\int e^{mx} \cos nx dx = \frac{e^{mx}(m \cos nx + n \sin mx)}{n^2 + m^2},$$

we obtain

$$L[\cos ax] = \int_0^\infty e^{-px} \cos ax dx = \frac{p}{p^2 + a^2}, \qquad p > 0.$$

(6) $f(x) = \sinh ax$, where a is a real constant.

Using the linearity property of the Laplace transform operator L, we obtain

$$L[\cosh ax] = L\left[\frac{e^{ax} + e^{-ax}}{2}\right] = \frac{1}{2} L[e^{ax}] + \frac{1}{2} L[e^{-ax}]$$

$$= \frac{1}{2}\left(\frac{1}{p - a} + \frac{1}{p + a}\right) = \frac{p}{p^2 - a^2}.$$

(7) $f(x) = x^k$, where $k > -1$.

By definition we have

$$L[x^k] = \int_0^\infty e^{-px} x^k dx.$$

Let $px = u$, then $dx = p^{-1} du, x^k = u^k/p^k$, and so

$$L[x^k] = \int_0^\infty e^{-px} x^k dx = \frac{1}{p^{k+1}} \int_0^\infty u^k e^{-u} du = \frac{\Gamma(k+1)}{p^{k+1}}.$$

Note that the integral defining the gamma function converges if and only if $k > -1$.

The following example illustrates the calculation of inverse Laplace transforms which is equally important in solving differential equations.

Example 9.3

Find

$$(a)\ L^{-1}\left[\frac{5}{p+2}\right], \qquad (b)\ L^{-1}\left[\frac{1}{p^s}\right], \quad s > 0.$$

Solution:

$$(a)\ L^{-1}\left[\frac{5}{p+2}\right] = 5L^{-1}\left[\frac{1}{p+2}\right].$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

Recall $L[e^{ax}] = 1/(p-a)$, hence $L^{-1}[1/(p-a)] = e^{ax}$. It follows that

$$L^{-1}\left[\frac{5}{p+2}\right] = 5L^{-1}\left[\frac{1}{p+2}\right] = 5e^{-2x}.$$

(*b*) Recall

$$L[x^k] = \int_0^\infty e^{-px}x^k dx = \frac{1}{p^{k+1}}\int_0^\infty u^k e^{-u}du = \frac{\Gamma(k+1)}{p^{k+1}}.$$

From this we have

$$L\left[\frac{x^k}{\Gamma(k+1)}\right] = \frac{1}{p^{k+1}},$$

hence

$$L^{-1}\left[\frac{1}{p^{k+1}}\right] = \frac{x^k}{\Gamma(k+1)}.$$

If we now let $k+1=s$, then

$$L^{-1}\left[\frac{1}{p^s}\right] = \frac{x^{s-1}}{\Gamma(s)}.$$

Shifting (or translation) theorems

In practical applications, we often meet functions multiplied by exponential factors. If we know the Laplace transform of a function, then multiplying it by an exponential factor does not require a new computation as shown by the following theorem.

The first shifting theorem

$$\text{If}\ \ L[f(x)]=F(p),\ p>b;\ \textit{then}\ \ L[e^{at}f(x)] = F(p-a),\ p>a+b.$$

Note that $F(p-a)$ denotes the function $F(p)$ 'shifted' a units to the right. Hence the theorem is called the shifting theorem.

The proof is simple and straightforward. By definition (9.1) we have

$$L[f(x)] = \int_0^\infty e^{-px}f(x)dx = F(p).$$

Then

$$L[e^{ax}f(x)] = \int_0^\infty e^{-px}\{e^{ax}f(x)\}dx = \int_0^\infty e^{-(p-a)x}f(x)dx = F(p-a).$$

The following examples illustrate the use of this theorem.

Example 9.4
Show that:

$$(a)\ \ L[e^{-ax}x^n] = \frac{n!}{(p+a)^{n+1}}, \qquad p > -a;$$

$$(b)\ \ L[e^{-ax}\sin bx] = \frac{b}{(p+a)^2 + b^2}, \qquad p > -a.$$

Solution: (a) Recall

$$L[x^n] = n!/p^{n+1}, \qquad p > 0;$$

the shifting theorem then gives

$$L[e^{-ax}x^n] = \frac{n!}{(p+a)^{n+1}}, \qquad p > -a.$$

(b) Since

$$L[\sin ax] = \frac{a}{p^2 + a^2},$$

it follows from the shifting theorem that

$$L[e^{-ax}\sin bx] = \frac{b}{(p+a)^2 + b^2}, \qquad p > -a.$$

Because of the relationship between Laplace transforms and inverse Laplace transforms, any theorem involving Laplace transforms will have a corresponding theorem involving inverse Lapace transforms. Thus

If $L^{-1}[F(p)] = f(x)$, *then* $L^{-1}[F(p-a)] = e^{ax}f(x)$.

The second shifting theorem

This second shifting theorem involves the shifting x variable and states that

Given $L[f(x)] = F(p)$, *where* $f(x) = 0$ *for* $x < 0$; *and if* $g(x) = f(x-a)$, *then*

$$L[g(x)] = e^{-ap}L[f(x)].$$

To prove this theorem, let us start with

$$F(p) = L[f(x)] = \int_0^\infty e^{-px}f(x)dx$$

from which it follows that

$$e^{-ap}F(p) = e^{-ap}L[f(x)] = \int_0^\infty e^{-p(x+a)}f(x)dx.$$

Let $u = x + a$, then

$$e^{-ap}F(p) = \int_0^\infty e^{-p(x+a)} f(x)dx = \int_0^\infty e^{-pu} f(u-a)du$$
$$= \int_0^a e^{-pu} 0\, du + \int_a^\infty e^{-pu} f(u-a)du$$
$$= \int_0^\infty e^{-pu} g(u)du = L[g(u)].$$

Example 9.5
Show that given

$$f(x) = \begin{cases} x & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases},$$

and if

$$g(x) = \begin{cases} 0, & \text{for} \quad x < 5 \\ x - 5, & \text{for} \quad x \geq 5 \end{cases}$$

then

$$L[g(x)] = e^{-5p}/p^2.$$

Solution: We first notice that

$$g(x) = f(x-5).$$

Then the second shifting theorem gives

$$L[g(x)] = e^{-5p}L[x] = e^{-5p}/p^2.$$

The unit step function

It is often possible to express various discontinuous functions in terms of the unit step function, which is defined as

$$U(x-a) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}.$$

Sometimes it is convenient to state the second shifting theorem in terms of the unit step function:

If $f(x) = 0$ *for* $x < 0$ *and* $L[f(x)] = F(p)$, *then*

$$L[U(x-a)f(x-a)] = e^{-ap}F(p).$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

The proof is straightforward:

$$L[U(x-a)f(x-a)] = \int_0^\infty e^{-px}U(x-a)f(x-)dx$$

$$= \int_0^a e^{-px}0\,dx + \int_a^\infty e^{-px}f(x-a)dx.$$

Let $x - a = u$, then

$$L[U(x-a)f(x-a)] = \int_a^\infty e^{-px}f(x-a)dx$$

$$= \int_a^\infty e^{-p(u+a)}f(u)du = e^{-ap}\int_a^\infty e^{-pu}f(u)du = e^{-ap}F(p).$$

The corresponding theorem involving inverse Laplace transforms can be stated as

$$\textit{If}\ \ f(x) = 0\ \ \textit{for}\ \ x < 0\ \ \textit{and}\ \ L^{-1}[F(p)] = f(x),\ \ \textit{then}$$
$$L^{-1}[e^{-ap}F(p)] = U(x-a)f(x-a).$$

Laplace transform of a periodic function

If $f(x)$ is a periodic function of period $P > 0$, that is, if $f(x+P) = f(x)$, then

$$L[f(x)] = \frac{1}{1-e^{-pP}}\int_0^P e^{-px}f(x)dx.$$

To prove this, we assume that the Laplace transform of $f(x)$ exists:

$$L[f(x)] = \int_0^\infty e^{-px}f(x)dx = \int_0^P e^{-px}f(x)dx + \int_P^{2P} e^{-px}f(x)dx$$
$$+ \int_{2P}^{3P} e^{-px}f(x)dx + \cdots.$$

On the right hand side, let $x = u + P$ in the second integral, $x = u + 2P$ in the third integral, and so on, we then have

$$L[f(x)] = \int_0^P e^{-px}f(x)dx + \int_0^P e^{-p(u+P)}f(u+P)du$$
$$+ \int_0^P e^{-p(u+2P)}f(u+2P)du + \cdots.$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

But $f(u+P) = f(u)$, $f(u+2P) = f(u)$, etc. Also, let us replace the dummy variable u by x, then the above equation becomes

$$
\begin{aligned}
L[f(x)] &= \int_0^P e^{-px} f(x)dx + \int_0^P e^{-p(x+P)} f(x)dx + \int_0^P e^{-p(x+2P)} f(x)dx + \cdots \\
&= \int_0^P e^{-px} f(x)dx + e^{-pP} \int_0^P e^{-px} f(x)dx + e^{-2pP} \int_0^P e^{-px} f(x)dx + \cdots \\
&= (1 + e^{-pP} + e^{-2pP} + \cdots) \int_0^P e^{-px} f(x)dx \\
&= \frac{1}{1 - e^{-pP}} \int_0^P e^{-px} f(x)dx.
\end{aligned}
$$

Laplace transforms of derivatives

If $f(x)$ is a continuous for $x \geq 0$, and $f'(x)$ is piecewise continuous in every finite interval $0 \leq x \leq k$, and if $|f(x)| \leq Me^{bx}$ (that is, $f(x)$ is of exponential order), then

$$L[f'(x)] = pL[f(x)] - f(0), \quad p > b.$$

We may employ integration by parts to prove this result:

$$\int udv = uv - \int vdu \qquad \text{with} \qquad u = e^{-px}, \quad \text{and} \quad dv = f'(x)dx;$$

$$L[f'(x)] = \int_0^\infty e^{-px} f'(x)dx = [e^{-px} f(x)]_0^\infty - \int_0^\infty (-p)e^{-px} f(x)dx.$$

Since $|f(x)| \leq Me^{bx}$ for sufficiently large x, then $|f(x)e^{-px}| \leq Me^{(b-p)}$ for sufficiently large x. If $p > b$, then $Me^{(b-p)} \to 0$ as $x \to \infty$; and $e^{-px} f(x) \to 0$ as $x \to \infty$. Next, $f(x)$ is continuous at $x = 0$, and so $e^{-px} f(x) \to f(0)$ as $x \to 0$. Thus, the desired result follows:

$$L[f'(x)] = pL[f(x)] - f(0), \quad p > b.$$

This result can be extended as follows:

If $f(x)$ is such that $f^{(n-1)}(x)$ is continuous and $f^{(n)}(x)$ piecewise continuous in every interval $0 \leq x \leq k$ and furthermore, if $f(x), f'(x), \ldots, f^{(n)}(x)$ are of exponential order for $0 > k$, then

$$L[f^{(n)}(x)] = p^n L[f(x)] - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Example 9.6

Solve the initial value problem:

$y'' + y = 0; y(0) = y'(0) = 0$, and $f(t) = 0$ for $t < 0$ but $f(t) = 1$ for $t \geq 0$.

Solution: Note that $y' = dy/dt$. We know how to solve this simple differential equation, but as an illustration we now solve it using Laplace transforms. Taking both sides of the equation we obtain

$$L[y''] + L[y] = L[1], \quad (L[f] = L[1]).$$

Now

$$\begin{aligned} L[y''] &= pL[y'] - y'(0) = p\{pL[y] - y(0)\} - y'(0) \\ &= p^2 L[y] - py(0) - y'(0) \\ &= p^2 L[y] \end{aligned}$$

and

$$L[1] = 1/p.$$

The transformed equation then becomes

$$p^2 L[y] + L[y] = 1/p$$

or

$$L[y] = \frac{1}{p(p^2+1)} = \frac{1}{p} - \frac{p}{p^2+1},$$

therefore

$$y = L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{p}{p^2+1}\right].$$

We find from Eqs. (9.6) and (9.10) that

$$L^{-1}\left[\frac{1}{p}\right] = 1 \quad \text{and} \quad L^{-1}\left[\frac{p}{p^2+1}\right] = \cos t.$$

Thus, the solution of the initial problem is

$$y = 1 - \cos t \quad \text{for} \quad t \geq 0, \qquad y = 0 \quad \text{for} \quad t < 0.$$

Laplace transforms of functions defined by integrals

If $g(x) = \int_0^x f(u)du$, *and if* $L[f(x)] = F(p)$, *then* $L[g(x)] = F(p)/p$.

Similarly, if $L^{-1}[F(p)] = f(x)$, then $L^{-1}[F(p)/p] = g(x)$.

It is easy to prove this. If $g(x) = \int_0^x f(u)du$, then $g(0) = 0, g'(x) = f(x)$. Taking Laplace transform, we obtain

$$L[g'(x)] = L[f(x)]$$

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

but

$$L[g'(x)] = pL[g(x)] - g(0) = pL[g(x)]$$

and so

$$pL[g(x)] = L[f(x)], \quad \text{or} \quad L[g(x)] = \frac{1}{p}L[f(x)] = \frac{F(p)}{p}.$$

From this we have

$$L^{-1}[F(p)/p] = g(x).$$

Example 9.7
If $g(x) = \int_0^u \sin au\, du$, then

$$L[g(x)] = L\left[\int_0^u \sin au\, du\right] = \frac{1}{p}L[\sin au] = \frac{a}{p(p^2+a^2)}.$$

A note on integral transformations

A Laplace transform is one of the integral transformations. The integral transformation $T[f(x)]$ of a function $f(x)$ is defined by the integral equation

$$T[f(x)] = \int_a^b f(x)K(p,x)dx = F(p), \tag{9.6}$$

where $K(p,x)$, a known function of p and x, is called the kernel of the transformation. In the application of integral transformations to the solution of boundary-value problems, we have so far made use of five different kernels:

Laplace transform: $K(p,x) = e^{-px}$, and $a = 0, b = \infty$:

$$L[f(x)] = \int_0^\infty e^{-px}f(x)dx = (p).$$

Fourier sine and cosine transforms: $K(p,x) = \sin px$ or $\cos px$, and $a = 0, b = \infty$:

$$F[f(x)] = \int_0^\infty f(x)\begin{cases} \sin(px) \\ \cos(px) \end{cases} dx = F(p).$$

Complex Fourier transform: $K(p,x) = e^{ipx}$, and $a = -\infty$, $b = -\infty$:

$$F[f(x)] = \int_{-\infty}^{\infty} e^{ipx}f(x)dx = F(p).$$

Hankel transform: $K(p,x) = xJ_n(px), a = 0, b = \infty$, where $J_n(px)$ is the Bessel function of the first kind of order n:

$$H[f(x)] = \int_0^\infty f(x)xJ_n(x)dx = F(p).$$

Mellin transform: $K(p,x) = x^{p-1}$, and $a = 0, b = \infty$:

$$M[f(x)] = \int_0^\infty f(x)x^{p-1}dx = F(p).$$

The Laplace transform has been the subject of this chapter, and the Fouier transform was treated in Chapter 4. It is beyond the scope of this book to include Hankel and Mellin transformations.

Problems

9.1 Show that:
(a) e^{t^2} is not of exponential order as $x \to \infty$.
(b) $\sin e^{t^2}$ is of exponential order as $x \to \infty$.

9.2 Show that:

(a) $L[\sinh ax] = \dfrac{a}{p^2 - a^2}, \qquad p > 0.$

(b) $L[3x^4 - 2x^{3/2} + 6] = \dfrac{72}{p^5} - \dfrac{3\sqrt{\pi}}{2p^{5/2}} + \dfrac{6}{p}.$

(c) $L[\sin x \cos x] = 1/(p^2 + 4)$.

(d) If

$$f(x) = \begin{cases} x, & 0 < x < 4 \\ 5, & x > 4 \end{cases},$$

then

$$L[f(x) = \frac{1}{p^2} + \frac{e^{-4p}}{p} - \frac{e^{-4p}}{p^2}.$$

9.4 Show that $L[U(x-a)] = e^{-ap}/p, \ p > 0$.

9.5 Find the Laplace transform of $H(x)$, where

$$H(x) = \begin{cases} x, & 0 < x < 4 \\ 5, & x > 4 \end{cases}.$$

9.5 Let $f(x)$ be the rectified sine wave of period $P = 2\pi$:

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \pi \le x < 2\pi \end{cases}.$$

Find the Laplace transform of $f(x)$.

https://doi.org/10.1017/CBO9780511755781.010 Published online by Cambridge University Press

9.6 Find

$$L^{-1}\left[\frac{15}{p^2+4p+13}\right].$$

9.7 Prove that if $f'(x)$ is continuous and $f''(x)$ is piecewise continuous in every finite interval $0 \le x \le k$ and if $f(x)$ and $f'(x)$ are of exponential order for $x > k$, then

$$L[f'''(x)] = p^2 L[f(x)] - pf(0) - f'(0).$$

(Hint: Use (9.19) with $f'(x)$ in place of $f(x)$ and $f''(x)$ in place of $f'(x)$.)

9.8 Solve the initial problem $y''(t) + \beta^2 y(t) = A\sin\omega t;\ y(0) = 1,\ y'(0) = 0.$

9.9 Solve the initial problem $y'''(t) - y'(t) = \sin t$ subject to

$$y(0) = 2, \qquad y(0) = 0, \qquad y''(0) = 1.$$

9.10 Solve the linear simultaneous differential equation with constant coefficients

$$y'' + 2y - x = 0,$$
$$x'' + 2x - y = 0,$$

subject to $x(0) = 2,\ y(0) = 0$, and $x'(0) = y'(0) = 0$, where x and y are the dependent variables and t is the independent variable.

9.11 Find

$$L\left[\int_0^\infty \cos au\, du\right].$$

9.12. Prove that if $L[f(x)] = F(p)$ then

$$L[f(ax)] = \frac{1}{a}F\left(\frac{p}{a}\right).$$

Similarly if $L^{-1}[F(p)] = f(x)$ then

$$L^{-1}\left[F\left(\frac{p}{a}\right)\right] = af(ax).$$