10

Partial differential equations

We have met some partial differential equations in previous chapters. In this chapter we will study some elementary methods of solving partial differential equations which occur frequently in physics and in engineering. In general, the solution of partial differential equations presents a much more difficult problem than the solution of ordinary differential equations. A complete discussion of the general theory of partial differential equations is well beyond the scope of this book. We therefore limit ourselves to a few solvable partial differential equations that are of physical interest.

Any equation that contains an unknown function of two or more variables and its partial derivatives with respect to these variables is called a partial differential equation, the order of the equation being equal to the order of the highest partial derivatives present. For example, the equations

$$3y^2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2u, \quad \frac{\partial^2 u}{\partial x\partial y}=2x-y$$

are typical partial differential equations of the first and second orders, respectively, x and y being independent variables and $u(x, y)$ the function to be found. These two equations are linear, because both u and its derivatives occur only to the first order and products of u and its derivatives are absent. We shall not consider non-linear partial differential equations.

We have seen that the general solution of an ordinary differential equation contains arbitrary constants equal in number to the order of the equation. But the general solution of a partial differential equation contains arbitrary functions (equal in number to the order of the equation). After the particular choice of the arbitrary functions is made, the general solution becomes a particular solution.

The problem of finding the solution of a given differential equation subject to given initial conditions is called a boundary-value problem or an initial-value

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problem. We have seen already that such problems often lead to eigenvalue problems.

Linear second-order partial differential equations

Many physical processes can be described to some degree of accuracy by linear second-order partial differential equations. For simplicity, we shall restrict our discussion to the second-order linear partial differential equation in two independent variables, which has the general form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \qquad (10.1)$$

where $A, B, C, \ldots, G$ may be dependent on variables x and y.

If G is a zero function, then Eq. (10.1) is called homogeneous; otherwise it is said to be non-homogeneous. If $u_1, u_2, \ldots, u_n$ are solutions of a linear homogeneous partial differential equation, then $c_1u_1 + c_2u_2 + \cdots + c_nu_n$ is also a solution, where $c_1, c_2, \ldots$ are constants. This is known as the superposition principle; it does not apply to non-linear equations. The general solution of a linear non-homogeneous partial differential equation is obtained by adding a particular solution of the non-homogeneous equation to the general solution of the homogeneous equation.

The homogeneous form of Eq. (10.1) resembles the equation of a general conic:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

We thus say that Eq. (10.1) is of

$$\left.\begin{array}{c}\text{elliptic}\\ \text{hyperbolic}\\ \text{parabolic}\end{array}\right\}\text{type} \qquad \text{when}\left\{\begin{array}{l}B^2 - 4AC < 0\\ B^2 - 4AC > 0\\ B^2 - 4AC = 0\end{array}\right. .$$

For example, according to this classification the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is of elliptic type ($A = C = 1, B = D = E = F = G = 0$), and the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha^2\frac{\partial^2 u}{\partial y^2} = 0 \quad (\alpha \text{ is a real constant})$$

is of hyperbolic type. Similarly, the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial y} = 0 \quad (\alpha \text{ is a real constant})$$

is of parabolic type.

We now list some important linear second-order partial differential equations that are of physical interest and we have seen already:

(1) Laplace's equation:

$$\nabla^2 u = 0, \tag{10.2}$$

where ∇^2 is the Laplacian operator. The function u may be the electrostatic potential in a charge-free region. It may be the gravitational potential in a region containing no matter or the velocity potential for an incompressible fluid with no sources or sinks.

(2) Poisson's equation:

$$\nabla^2 u = \rho(x, y, z), \tag{10.3}$$

where the function $\rho(x, y, z)$ is called the source density. For example, if u represents the electrostatic potential in a region containing charges, then ρ is proportional to the electrical charge density. Similarly, for the gravitational potential case, ρ is proportional to the mass density in the region.

(3) Wave equation:

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \tag{10.4}$$

transverse vibrations of a string, longitudinal vibrations of a beam, or propagation of an electromagnetic wave all obey this same type of equation. For a vibrating string, u represents the displacement from equilibrium of the string; for a vibrating beam, u is the longitudinal displacement from the equilibrium. Similarly, for an electromagnetic wave, u may be a component of electric field $\mathbf{E}$ or magnetic field $\mathbf{H}$.

(4) Heat conduction equation:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u, \tag{10.5}$$

where u is the temperature in a solid at time t. The constant α is called the diffusivity and is related to the thermal conductivity, the specific heat capacity, and the mass density of the object. Eq. (10.5) can also be used as a diffusion equation: u is then the concentration of a diffusing substance.

It is obvious that Eqs. (10.2)–(10.5) all are homogeneous linear equations with constant coefficients.

Example 10.1

Laplace's equation: arises in almost all branches of analysis. A simple example can be found from the motion of an incompressible fluid. Its velocity $\mathbf{v}(x, y, z, t)$ and the fluid density $\rho(x, y, z, t)$ must satisfy the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

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If ρ is constant we then have

$$\nabla \cdot \mathbf{v} = 0.$$

If, furthermore, the motion is irrotational, the velocity vector can be expressed as the gradient of a scalar function V:

$$\mathbf{v} = -\nabla V,$$

and the equation of continuity becomes Laplace's equation:

$$\nabla \cdot \mathbf{v} = \nabla \cdot (-\nabla V) = 0, \quad \text{or} \quad \nabla^2 V = 0.$$

The scalar function V is called the velocity potential.

Example 10.2

Poisson's equation: The electrostatic field provides a good example of Poisson's equation. The electric force between any two charges q and q' in a homogeneous isotropic medium is given by Coulomb's law

$$\mathbf{F} = C\frac{qq'}{r^2}\hat{r},$$

where r is the distance between the charges, and $\hat{r}$ is a unit vector in the direction of the force. The constant C determines the system of units, which is not of interest to us; thus we leave C as it is.

An electric field $\mathbf{E}$ is said to exist in a region if a stationary charge q' in that region experiences a force $\mathbf{F}$:

$$\mathbf{E} = \lim_{q' \to 0}(\mathbf{F}/q').$$

The $\lim_{q' \to 0}$ guarantees that the test charge q' will not alter the charge distribution that existed prior to the introduction of the test charge q'. From this definition and Coulomb's law we find that the electric field at a point r distant from a point charge is given by

$$\mathbf{E} = C\frac{q}{r^2}\hat{r}.$$

Taking the curl on both sides we get

$$\nabla \times \mathbf{E} = 0,$$

which shows that the electrostatic field is a conservative field. Hence a potential function ϕ exists such that

$$\mathbf{E} = -\nabla\phi.$$

Taking the divergence of both sides

$$\nabla \cdot (\nabla\phi) = -\nabla \cdot \mathbf{E}$$

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or

$$\nabla^2 \phi = -\nabla \cdot \mathbf{E}.$$

$\nabla \cdot \mathbf{E}$ is given by Gauss' law. To see this, consider a volume τ containing a total charge q. Let $d\mathbf{s}$ be an element of the surface S which bounds the volume τ. Then

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = Cq \iint_S \frac{\hat{r} \cdot d\mathbf{s}}{r^2}.$$

The quantity $\hat{r} \cdot d\mathbf{s}$ is the projection of the element area ds on a plane perpendicular to $\mathbf{r}$. This projected area divided by r^2 is the solid angle subtended by ds, which is written $d\Omega$. Thus, we have

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = Cq \iint_S \frac{\hat{r} \cdot d\mathbf{s}}{r^2} = Cq \iint_S d\Omega = 4\pi Cq.$$

If we write q as

$$q = \iiint_\tau \rho dV,$$

where ρ is the charge density, then

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi C \iiint_\tau \rho dV.$$

But (by the divergence theorem)

$$\iint_S \mathbf{E} \cdot d\mathbf{s} = \iiint_\tau \nabla \cdot \mathbf{E} dV.$$

Substituting this into the previous equation, we obtain

$$\iiint_\tau \nabla \cdot \mathbf{E} dV = 4\pi C \iiint_\tau \rho dV$$

or

$$\iiint_\tau (\nabla \cdot \mathbf{E} - 4\pi C\rho) dV = 0.$$

This equation must be valid for all volumes, that is, for any choice of the volume τ. Thus, we have Gauss' law in differential form:

$$\nabla \cdot \mathbf{E} = 4\pi C\rho.$$

Substituting this into the equation $\nabla^2 \phi = -\nabla \cdot \mathbf{E}$, we get

$$\nabla^2 \phi = -4\pi C\rho,$$

which is Poisson's equation. In the Gaussian system of units, $C = 1$; in the SI system of units, $C = 1/4\pi\varepsilon_0$, where the constant ε_0 is known as the permittivity of free space. If we use SI units, then

$$\nabla^2 \phi = -\rho/\varepsilon_0.$$

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In the particular case of zero charge density it reduces to Laplace's equation,

$$\nabla^2 \phi = 0.$$

In the following sections, we shall consider a number of problems to illustrate some useful methods of solving linear partial differential equations. There are many methods by which homogeneous linear equations with constant coefficients can be solved. The following are commonly used in the applications.

(1) General solutions: In this method we first find the general solution and then that particular solution which satisfies the boundary conditions. It is always satisfying from the point of view of a mathematician to be able to find general solutions of partial differential equations; however, general solutions are difficult to find and such solutions are sometimes of little value when given boundary conditions are to be imposed on the solution. To overcome this difficulty it is best to find a less general type of solution which is satisfied by the type of boundary conditions to be imposed. This is the method of separation of variables.

(2) Separation of variables: The method of separation of variables makes use of the principle of superposition in building up a linear combination of individual solutions to form a solution satisfying the boundary conditions. The basic approach of this method in attempting to solve a differential equation (in, say, two dependent variables x and y) is to write the dependent variable $u(x, y)$ as a product of functions of the separate variables $u(x, y) = X(x)Y(y)$. In many cases the partial differential equation reduces to ordinary differential equations for X and Y.

(3) Laplace transform method: We first obtain the Laplace transform of the partial differential equation and the associated boundary conditions with respect to one of the independent variables, and then solve the resulting equation for the Laplace transform of the required solution which can be found by taking the inverse Laplace transform.

Solutions of Laplace's equation: separation of variables

(1) Laplace's equation in two dimensions (x, y): If the potential ϕ is a function of only two rectangular coordinates, Laplace's equation reads

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

It is possible to obtain the general solution to this equation by means of a transformation to a new set of independent variables:

$$\xi = x + iy, \quad \eta = x - iy,$$

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where I is the unit imaginary number. In terms of these we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\ &= \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\frac{\partial \eta}{\partial x} \\ &= \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2}.\end{aligned}$$

Similarly, we have

$$\frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial^2}{\partial \eta^2}$$

and Laplace's equation now reads

$$\nabla^2 \phi = 4\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0.$$

Clearly, a very general solution to this equation is

$$\phi = f_1(\xi) + f_2(\eta) = f_1(x + iy) + f_2(x - iy),$$

where f_1 and f_2 are arbitrary functions which are twice differentiable. However, it is a somewhat difficult matter to choose the functions f_1 and f_2 such that the equation is, for example, satisfied inside a square region defined by the lines $x = 0, x = a, y = 0, y = b$ and such that ϕ takes prescribed values on the boundary of this region. For many problems the method of separation of variables is more satisfactory. Let us apply this method to Laplace's equation in three dimensions.

(2) Laplace's equation in three dimensions (x, y, z): Now we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{10.6}$$

We make the assumption, justifiable by its success, that $\phi(x, y, z)$ may be written as the product

$$\phi(x, y, z) = X(x)Y(y)Z(z).$$

Substitution of this into Eq. (10.6) yields, after division by ϕ,

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2}. \tag{10.7}$$

The left hand side of Eq. (10.7) is a function of x and y, while the right hand side is a function of z alone. If Eq. (10.7) is to have a solution at all, each side of the equation must be equal to the same constant, say k_3^2. Then Eq. (10.7) leads to

$$\frac{d^2Z}{dz^2} + k_3^2 Z = 0, \tag{10.8}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} + k_3^2. \tag{10.9}$$

The left hand side of Eq. (10.9) is a function of x only, while the right hand side is a function of y only. Thus, each side of the equation must be equal to a constant, say k_1^2. Therefore

$$\frac{d^2X}{dx^2} + k_1^2 X = 0, \tag{10.10}$$

$$\frac{d^2Y}{dy^2} + k_2^2 Y = 0, \tag{10.11}$$

where

$$k_2^2 = k_1^2 - k_3^2.$$

The solution of Eq. (10.10) is of the form

$$X(x) = a(k_1)e^{k_1 x}, \quad k_1 \neq 0, \quad -\infty < k_1 < \infty$$

or

$$X(x) = a(k_1)e^{k_1 x} + a'(k_1)e^{-k_1 x}, \quad k_1 \neq 0, \quad 0 < k_1 < \infty. \tag{10.12}$$

Similarly, the solutions of Eqs. (10.11) and (10.8) are of the forms

$$Y(y) = b(k_2)e^{k_2 y} + b'(k_2)e^{-k_2 y}, \quad k_2 \neq 0, \quad 0 < k_2 < \infty, \tag{10.13}$$

$$Z(z) = c(k_3)e^{k_3 z} + c'(k_3)e^{-k_3 z}, \quad k_3 \neq 0, \quad 0 < k_3 < \infty. \tag{10.14}$$

Hence

$$\phi = [a(k_1)e^{k_1 x} + a'(k_1)e^{-k_1 x}][b(k_2)e^{k_2 y} + b'(k_2)e^{-k_2 y}][c(k_3)e^{k_3 z} + c'(k_3)e^{-k_3 z}],$$

and the general solution of Eq. (10.6) is obtained by integrating the above equation over all the permissible values of the k_i $(i = 1, 2, 3)$.

In the special case when $k_i = 0$ $(i = 1, 2, 3)$, Eqs. (10.8), (10.10), and (10.11) have solutions of the form

$$X_i(x_i) = a_i x_i + b_i,$$

where $x_1 = x$, and $X_1 = X$ etc.

Let us now apply the above result to a simple problem in electrostatics: that of finding the potential ϕ at a point P a distance h from a uniformly charged infinite plane in a dielectric of permittivity ε. Let σ be the charge per unit area of the plane, and take the origin of the coordinates in the plane and the x-axis perpendicular to the plane. It is evident that ϕ is a function of x only. There are two types of solutions, namely:

$$\phi(x) = a(k_1)e^{k_1 x} + a'(k_1)e^{-k_1 x},$$
$$\phi(x) = a_1 x + b_1;$$

the boundary conditions will eliminate the unwanted one. The first boundary condition is that the plane is an equipotential, that is, $\phi(0) =$ constant, and the second condition is that $E = -\partial\phi/\partial x = \sigma/2\varepsilon$. Clearly, only the second type of solution satisfies both the boundary conditions. Hence $b_1 = \phi(0), a_1 = -\sigma/2\varepsilon$, and the solution is

$$\phi(x) = -\frac{\sigma}{2\varepsilon}x + \phi(0).$$

(3) Laplace's equation in cylindrical coordinates (ρ, φ, z): The cylindrical coordinates are shown in Fig. 10.1, where

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \rho^2 &= x^2 + y^2 \\ \varphi &= \tan^{-1}(y/x) \\ z &= z. \end{aligned}\right.$$

Laplace's equation now reads

$$\nabla^2\phi(\rho, \varphi, z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial^2 z^2} = 0. \qquad (10.15)$$

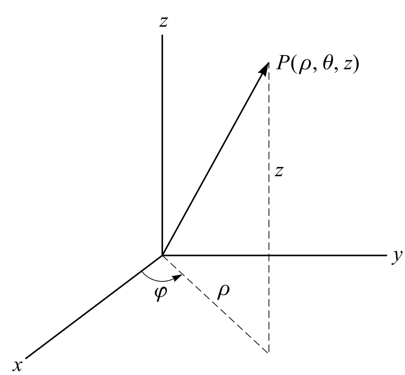

Figure 10.1. Cylindrical coordinates.

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We assume that

$$\phi(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z). \tag{10.16}$$

Substitution into Eq. (10.15) yields, after division by ϕ,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{10.17}$$

Clearly, both sides of Eq. (10.17) must be equal to a constant, say $-k^2$. Then

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \quad \text{or} \quad \frac{d^2Z}{dz^2} - k^2Z = 0 \tag{10.18}$$

and

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2} = -k^2$$

or

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + k^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}.$$

Both sides of this last equation must be equal to a constant, say α^2. Hence

$$\frac{d^2\Phi}{d\varphi^2} + \alpha^2\Phi = 0, \tag{10.19}$$

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left(k^2 - \frac{\alpha^2}{\rho^2}\right)R = 0. \tag{10.20}$$

Equation (10.18) has for solutions

$$Z(z) = \begin{cases} c(k)e^{kz} + c'(k)e^{-kz}, & k \neq 0, \quad 0 < k < \infty, \\ c_1z + c_2, & k = 0, \end{cases} \tag{10.21}$$

where c and c' are arbitrary functions of k and c_1 and c_2 are arbitrary constants.

Equation (10.19) has solutions of the form

$$\Phi(\varphi) = \begin{cases} a(\alpha)e^{i\alpha\varphi}, & \alpha \neq 0, \quad -\infty < \alpha < \infty, \\ b\varphi + b', & \alpha = 0. \end{cases}$$

That the potential must be single-valued requires that $\Phi(\varphi) = \Phi(\varphi + 2n\pi)$, where n is an integer. It follows from this that α must be an integer or zero and that $b = 0$. Then the solution $\Phi(\varphi)$ becomes

$$\Phi(\varphi) = \begin{cases} a(\alpha)e^{i\alpha\varphi} + a'(\alpha)e^{-i\alpha\varphi}, & \alpha \neq 0, \quad \alpha = \text{integer}, \\ b', & \alpha = 0. \end{cases} \tag{10.22}$$

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In the special case $k = 0$, Eq. (10.20) has solutions of the form

$$R(\rho) = \begin{cases} d(\alpha)\rho^{\alpha} + d'(\alpha)\rho^{-\alpha}, & \alpha \neq 0, \\ f \ln \rho + g, & \alpha = 0. \end{cases} \tag{10.23}$$

When $k \neq 0$, a simple change of variable can put Eq. (10.20) in the form of Bessel's equation. Let $x = k\rho$, then $dx = kd\rho$ and Eq. (10.20) becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right)R = 0, \tag{10.24}$$

the well-known Bessel's equation (Eq. (7.71)). As shown in Chapter 7, $R(x)$ can be written as

$$R(x) = AJ_{\alpha}(x) + BJ_{-\alpha}(x), \tag{10.25}$$

where A and B are constants, and $J_{\alpha}(x)$ is the Bessel function of the first kind. When α is not an integer, J_{α} and $J_{-\alpha}$ are independent. But when α is an integer, $J_{-\alpha}(x) = (-1)^n J_{\alpha}(x)$, thus J_{α} and $J_{-\alpha}$ are linearly dependent, and Eq. (10.25) cannot be a general solution. In this case the general solution is given by

$$R(x) = A_1 J_{\alpha}(x) + B_1 Y_{\alpha}(x), \tag{10.26}$$

where A_1 and B_2 are constants; $Y_{\alpha}(x)$ is the Bessel function of the second kind of order α or Neumann's function of order $\alpha N_{\alpha}(x)$.

The general solution of Eq. (10.20) when $k \neq 0$ is therefore

$$R(\rho) = p(\alpha)J_{\alpha}(k\rho) + q(\alpha)Y_{\alpha}(k\rho), \tag{10.27}$$

where p and q are arbitrary functions of α. Then these functions are also solutions:

$$H_{\alpha}^{(1)}(k\rho) = J_{\alpha}(k\rho) + iY_{\alpha}(k\rho), H_{\alpha}^{(2)}(k\rho) = J_{\alpha}(k\rho) - iY_{\alpha}(k\rho).$$

These are the Hankel functions of the first and second kinds of order α, respectively.

The functions J_{α}, Y_{α} (or N_{α}), and $H_{\alpha}^{(1)}$, and $H_{\alpha}^{(2)}$ which satisfy Eq. (10.20) are known as cylindrical functions of integral order α and are denoted by $Z_{\alpha}(k\rho)$, which is not the same as $Z(z)$. The solution of Laplace's equation (10.15) can now be written

$$\phi(\rho, \varphi, z) \begin{cases} = (c_1 z + b)(f \ln \rho + g), & k = 0, \quad \alpha = 0, \\ = (c_1 z + b)[d(\alpha)\rho^{\alpha} + d'(\alpha)\rho^{-\alpha}][a(\alpha)e^{i\alpha\varphi} + a'(\alpha)e^{-i\alpha\varphi}], & \\ & k = 0, \quad \alpha \neq 0, \\ = [c(k)e^{kz} + c'(k)e^{-kz}]Z_0(k\rho), & k \neq 0, \quad \alpha = 0, \\ = [c(k)e^{kz} + c'(k)e^{-kz}]Z_{\alpha}(k\rho)[a(\alpha)e^{i\alpha\varphi} + a'(\alpha)e^{-i\alpha\varphi}], & k \neq 0, \quad \alpha \neq 0. \end{cases}$$

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Let us now apply the solutions of Laplace's equation in cylindrical coordinates to an infinitely long cylindrical conductor with radius l and charge per unit length λ. We want to find the potential at a point P a distance $\rho > l$ from the axis of the cylindrical. Take the origin of the coordinates on the axis of the cylinder that is taken to be the z-axis. The surface of the cylinder is an equipotential:

$$\phi(l) = \text{const.} \quad \text{for } r = l \text{ and all } \varphi \text{ and } z.$$

The secondary boundary condition is that

$$E = -\partial\phi/\partial\rho = \lambda/2\pi l\varepsilon \quad \text{for } r = l \text{ and all } \varphi \text{ and } z.$$

Of the four types of solutions to Laplace's equation in cylindrical coordinates listed above only the first can satisfy these two boundary conditions. Thus

$$\phi(\rho) = b(f \ln\rho + g) = -\frac{\lambda}{2\pi\varepsilon}\ln\frac{\rho}{l} + \phi(a).$$

(4) Laplace's equation in spherical coordinates (r, θ, φ): The spherical coordinates are shown in Fig. 10.2, where

$$x = r\sin\theta\cos\varphi,$$
$$y = r\sin\theta\sin\varphi,$$
$$z = r\cos\varphi.$$

Laplace's equation now reads

$$\nabla^2\phi(r,\theta,\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0. \tag{10.28}$$

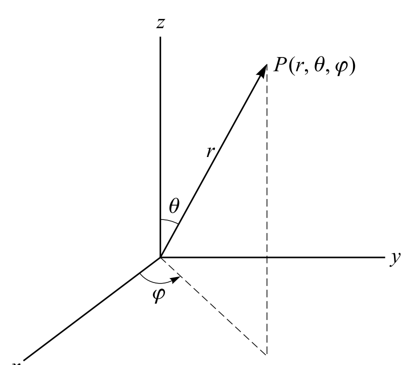

Figure 10.2. Spherical coordinates.

Again, assume that

$$\phi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi). \tag{10.29}$$

Substituting into Eq. (10.28) and dividing by ϕ we obtain

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}.$$

For a solution, both sides of this last equation must be equal to a constant, say m^2. Then we have two equations

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0, \tag{10.30}$$

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2;$$

the last equation can be rewritten as

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right).$$

Again, both sides of the last equation must be equal to a constant, say $-\beta$. This yields two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \beta, \tag{10.31}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\beta.$$

By a simple substitution: $x = \cos\theta$, we can put the last equation in a more familiar form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left(\beta - \frac{m^2}{1-x^2}\right)P = 0 \tag{10.32}$$

or

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[\beta - \frac{m^2}{1-x^2}\right]P = 0, \tag{10.32a}$$

where we have set $P(x) = \Theta(\theta)$.

You may have already noticed that Eq. (10.32) is very similar to Eq. (10.25), the associated Legendre equation. Let us take a close look at this resemblance. In Eq. (10.32), the points $x = \pm 1$ are regular singular points of the equation. Let us first study the behavior of the solution near point $x = 1$; it is convenient to bring this regular singular point to the origin, so we make the substitution $u = 1 - x, U(u) = P(x)$. Then Eq. (10.32) becomes

$$\frac{d}{du}\left[u(2-u)\frac{dU}{du}\right] + \left[\beta - \frac{m^2}{u(2-u)}\right]U = 0.$$

When we solve this equation by a power series: $U = \sum_{n=0}^{\infty} a_n u^{n+\rho}$, we find that the indicial equation leads to the values $\pm m/2$ for ρ. For the point $x = -1$, we make the substitution $v = 1 + x$, and then solve the resulting differential equation by the power series method; we find that the indicial equation leads to the same values $\pm m/2$ for ρ.

Let us first consider the value $+m/2, m \geq 0$. The above considerations lead us to assume

$$P(x) = (1-x)^{m/2}(1+x)^{m/2}y(x) = (1-x^2)^{m/2}y(x), \qquad m \geq 0$$

as the solution of Eq. (10.32). Substituting this into Eq. (10.32) we find

$$(1-x^2)\frac{d^2y}{dx^2} - 2(m+1)x\frac{dy}{dx} + [\beta - m(m+1)]y = 0.$$

Solving this equation by a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\delta},$$

we find that the indicial equation is $\delta(\delta - 1) = 0$. Thus the solution can be written

$$y(x) = \sum_{n\,\text{even}} c_n x^n + \sum_{n\,\text{odd}} c_n x^n.$$

The recursion formula is

$$c_{n+2} = \frac{(n+m)(n+m+1) - \beta}{(n+1)(n+2)} c_n.$$

Now consider the convergence of the series. By the ratio test,

$$R_n = \left|\frac{c_n x^n}{c_{n-2}x^{n-2}}\right| = \left|\frac{(n+m)(n+m+1) - \beta}{(n+1)(n+2)}\right| \cdot |x|^2.$$

The series converges for $|x| < 1$, whatever the finite value of β may be. For $|x| = 1$, the ratio test is inconclusive. However, the integral test yields

$$\int_M \frac{(t+m)(t+m+1) - \beta}{(t+1)(t+2)}\,dt = \int_M \frac{(t+m)(t+m+1)}{(t+1)(t+2)}\,dt - \int_M \frac{\beta}{(t+1)(t+2)}\,dt$$

and since

$$\int_M \frac{(t+m)(t+m+1)}{(t+1)(t+2)}\,dt \to \infty \quad \text{as} \quad M \to \infty,$$

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the series diverges for $|x| = 1$. A solution which converges for all x can be obtained if either the even or odd series is terminated at the term in x^j. This may be done by setting β equal to

$$\beta = (j + m)(j + m + 1) = l(l + 1).$$

On substituting this into Eq. (10.32a), the resulting equation is

$$(1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[l(l + 1) - \frac{m^2}{1 - x^2}\right]P = 0,$$

which is identical to Eq. (7.25). Special solutions were studied there: they were written in the form $P_l^m(x)$ and are known as the associated Legendre functions of the first kind of degree l and order m, where l and m, take on the values $l = 0, 1, 2, \ldots,$ and $m = 0, 1, 2, \ldots, l$. The general solution of Eq. (10.32) for $m \geq 0$ is therefore

$$P(x) = \Theta(\theta) = a_l P_l^m(x). \tag{10.33}$$

The second solution of Eq. (10.32) is given by the associated Legendre function of the second kind of degree l and order m: $Q_l^m(x)$. However, only the associated Legendre function of the first kind remains finite over the range $-1 \leq x \leq 1$ (or $0 \leq \theta \leq 2\pi$).

Equation (10.31) for $R(r)$ becomes

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l + 1)R = 0. \tag{10.31a}$$

When $l \neq 0$, its solution is

$$R(r) = b(l)r^l + b'(l)r^{-l-1}, \tag{10.34}$$

and when $l = 0$, its solution is

$$R(r) = cr^{-1} + d. \tag{10.35}$$

The solution of Eq. (10.30) is

$$\Phi = \begin{cases} f(m)e^{im\varphi} + f'(l)e^{-im\varphi}, & m \neq 0, \quad \text{positive integer}, \\ g, & m = 0. \end{cases} \tag{10.36}$$

The solution of Laplace's equation (10.28) is therefore given by

$$\phi(r, \theta, \varphi) = \begin{cases} [br^l + b'r^{-l-1}]P_l^m(\cos\theta)[fe^{im\varphi} + f'e^{-im\varphi}], & l \neq 0, \quad m \neq 0, \\ [br^l + b'r^{-l-1}]P_l(\cos\theta), & l \neq 0, \quad m = 0, \\ [cr^{-1} + d]P_0(\cos\theta), & l = 0, \quad m = 0, \end{cases} \tag{10.37}$$

where $P_l = P_l^0$.

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We now illustrate the usefulness of the above result for an electrostatic problem having spherical symmetry. Consider a conducting spherical shell of radius a and charge σ per unit area. The problem is to find the potential $\phi(r, \theta, \varphi)$ at a point P a distance $r > a$ from the center of shell. Take the origin of coordinates to be at the center of the shell. As the surface of the shell is an equipotential, we have the first boundary condition

$$\phi(r) = \text{constant} = \phi(a) \quad \text{for } r = a \quad \text{and all } \theta \text{ and } \varphi. \tag{10.38}$$

The second boundary condition is that

$$\phi \to 0 \quad \text{for } r \to \infty \text{ and all } \theta \text{ and } \varphi. \tag{10.39}$$

Of the three types of solutions (10.37) only the last can satisfy the boundary conditions. Thus

$$\phi(r, \theta, \varphi) = (cr^{-1} + d)P_0(\cos\theta). \tag{10.40}$$

Now $P_0(\cos\theta) = 1$, and from Eq. (10.38) we have

$$\phi(a) = ca^{-1} + d.$$

But the boundary condition (10.39) requires that $d = 0$. Thus $\phi(a) = ca^{-1}$, or $c = a\phi(a)$, and Eq. (10.40) reduces to

$$\phi(r) = \frac{a\phi(a)}{r}. \tag{10.41}$$

Now

$$\phi(a)/a = E(a) = Q/4\pi a^2\varepsilon,$$

where ε is the permittivity of the dielectric in which the shell is embedded, $Q = 4\pi a^2\sigma$. Thus $\phi(a) = a\sigma/\varepsilon$, and Eq. (10.41) becomes

$$\phi(r) = \frac{\sigma a^2}{\varepsilon r}. \tag{10.42}$$

Solutions of the wave equation: separation of variables

We now use the method of separation of variables to solve the wave equation

$$\frac{\partial^2 u(x, t)}{\partial x^2} = v^{-2}\frac{\partial^2 u(x, t)}{\partial t^2}, \tag{10.43}$$

subject to the following boundary conditions:

$$u(0, t) = u(l, t) = 0, \quad t \geq 0, \tag{10.44}$$

$$u(x, 0) = f(t), \quad 0 \leq x \leq l, \tag{10.45}$$

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and

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x), \quad 0 \le x \le l, \tag{10.46}$$

where f and g are given functions.

Assuming that the solution of Eq. (10.43) may be written as a product

$$u(x,t) = X(x)T(t), \tag{10.47}$$

then substituting into Eq. (10.43) and dividing by XT we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2 T}\frac{d^2T}{dt^2}.$$

Both sides of this last equation must be equal to a constant, say $-b^2/v^2$. Then we have two equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{b^2}{v^2}, \tag{10.48}$$

$$\frac{1}{T}\frac{d^2T}{dt^2} = -b^2. \tag{10.49}$$

The solutions of these equations are periodic, and it is more convenient to write them in terms of trigonometric functions

$$X(x) = A\sin\frac{bx}{v} + B\cos\frac{bx}{v}, \qquad T(t) = C\sin bt + D\cos bt, \tag{10.50}$$

where A, B, C, and D are arbitrary constants, to be fixed by the boundary conditions. Equation (10.47) then becomes

$$u(x,t) = \left(A\sin\frac{bx}{v} + B\cos\frac{bx}{v}\right)(C\sin bt + D\cos bt). \tag{10.51}$$

The boundary condition $u(0,t) = 0 (t > 0)$ gives

$$0 = B(C\sin bt + D\cos bt)$$

for all t, which implies

$$B = 0. \tag{10.52}$$

Next, from the boundary condition $u(l,t) = 0 (t > 0)$ we have

$$0 = A\sin\frac{bl}{v}(C\sin bt + D\cos bt).$$

Note that $B = 0$ would make $u = 0$. However, the last equation can be satisfied for all t when

$$\sin\frac{bl}{v} = 0,$$

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which implies

$$b = \frac{n\pi v}{l}, \quad n = 1, 2, 3, \ldots. \tag{10.53}$$

Note that n cannot be equal to zero, because it would make $b = 0$, which in turn would make $u = 0$.

Substituting Eq. (10.53) into Eq. (10.51) we have

$$u_n(x, t) = \sin\frac{n\pi x}{l}\left(C_n \sin\frac{n\pi vt}{l} + D_n \cos\frac{n\pi vt}{l}\right), \qquad n = 1, 2, 3, \ldots. \tag{10.54}$$

We see that there is an infinite set of discrete values of b and that to each value of b there corresponds a particular solution. Any linear combination of these particular solutions is also a solution:

$$u_n(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(C_n \sin\frac{n\pi vt}{l} + D_n \cos\frac{n\pi vt}{l}\right). \tag{10.55}$$

The constants C_n and D_n are fixed by the boundary conditions (10.45) and (10.46).

Application of boundary condition (10.45) yields

$$f(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}. \tag{10.56}$$

Similarly, application of boundary condition (10.46) gives

$$g(x) = \frac{\pi v}{l}\sum_{n=1}^{\infty} nC_n \sin\frac{n\pi x}{l}. \tag{10.57}$$

The coefficients C_n and D_n may then be determined by the Fourier series method:

$$D_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \qquad C_n = \frac{2}{n\pi v}\int_0^l g(x)\sin\frac{n\pi x}{l}dx. \tag{10.58}$$

We can use the method of separation of variable to solve the heat conduction equation. We shall leave this as a home work problem.

In the following sections, we shall consider two more methods for the solution of linear partial differential equations: the method of Green's functions, and the method of the Laplace transformation which was used in Chapter 9 for the solution of ordinary linear differential equations with constant coefficients.

Solution of Poisson's equation. Green's functions

The Green's function approach to boundary-value problems is a very powerful technique. The field at a point caused by a source can be considered to be the total effect due to each "unit" (or elementary portion) of the source. If $G(x; x')$ is the field at a point x due to a unit point source at x', then the total field at x due to a distributed source $\rho(x')$ is the integral of $G\rho$ over the range of x' occupied by the source. The function $G(x; x')$ is the well-known Green's function. We now apply this technique to solve Poisson's equation for electric potential ϕ (Example 10.2)

$$\nabla^2\phi(\mathbf{r}) = -\frac{1}{\varepsilon}\rho(\mathbf{r}), \tag{10.59}$$

where ρ is the charge density and ε the permittivity of the medium, both are given.

By definition, Green's function $G(\mathbf{r}; \mathbf{r}')$ is the solution of

$$\nabla^2 G(\mathbf{r}; \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{10.60}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function.

Now, multiplying Eq. (10.60) by ϕ and Eq. (10.59) by G, and then subtracting, we find

$$\phi(\mathbf{r})\nabla^2 G(\mathbf{r}; \mathbf{r}') - G(\mathbf{r}; \mathbf{r}')\nabla^2\phi(\mathbf{r}) = \phi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') + \frac{1}{\varepsilon}G(\mathbf{r}; \mathbf{r}')\rho(\mathbf{r});$$

and on interchanging $\mathbf{r}$ and $\mathbf{r}'$,

$$\phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}'; \mathbf{r}) - G(\mathbf{r}'; \mathbf{r})\nabla'^2\phi(\mathbf{r}') = \phi(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}) + \frac{1}{\varepsilon}G(\mathbf{r}'; \mathbf{r})\rho(\mathbf{r}')$$

or

$$\phi(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}) = \phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}'; \mathbf{r}) - G(\mathbf{r}'; \mathbf{r})\nabla'^2\phi(\mathbf{r}') - \frac{1}{\varepsilon}G(\mathbf{r}'; \mathbf{r})\rho(\mathbf{r}'), \tag{10.61}$$

the prime on ∇ indicates that differentiation is with respect to the primed coordinates. Integrating this last equation over all $\mathbf{r}'$ within and on the surface S' which encloses all sources (charges) yields

$$\begin{aligned}\phi(\mathbf{r}) = &-\frac{1}{\varepsilon}\int G(\mathbf{r}; \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' \\ &+ \int [\phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}; \mathbf{r}') - G(\mathbf{r}; \mathbf{r}')\nabla'^2\phi(\mathbf{r}')]d\mathbf{r}',\end{aligned} \tag{10.62}$$

where we have used the property of the delta function

$$\int_{-\infty}^{+\infty} f(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')d\mathbf{r}' = f(\mathbf{r}).$$

We now use Green's theorem

$$\iiint f\nabla'^2\psi - \psi\nabla'^2 f)d\tau' = \iint (f\nabla'\psi - \psi\nabla' f)\cdot d\mathbf{S}$$

to transform the second term on the right hand side of Eq. (10.62) and obtain

$$\phi(\mathbf{r}) = -\frac{1}{\varepsilon}\int G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int [\phi(\mathbf{r}')\nabla' G(\mathbf{r};\mathbf{r}') - G(\mathbf{r};\mathbf{r}')\nabla'\phi(\mathbf{r}')]\cdot d\mathbf{S}' \tag{10.63}$$

or

$$\phi(\mathbf{r}) = -\frac{1}{\varepsilon}\int G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int \left[\phi(\mathbf{r}')\frac{\partial}{\partial n'}G(\mathbf{r};\mathbf{r}') - G(\mathbf{r};\mathbf{r}')\frac{\partial}{\partial n'}\phi(\mathbf{r}')\right]\cdot d\mathbf{S}', \tag{10.64}$$

where $\mathbf{n}'$ is the outward normal to dS'. The Green's function $G(\mathbf{r};\mathbf{r}')$ can be found from Eq. (10.60) subject to the appropriate boundary conditions.

If the potential ϕ vanishes on the surface S' or $\partial\phi/\partial n'$ vanishes, Eq. (10.64) reduces to

$$\phi(\mathbf{r}) = -\frac{1}{\varepsilon}\int G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'. \tag{10.65}$$

On the other hand, if the surface S' encloses no charge, then Poisson's equation reduces to Laplace's equation and Eq. (10.64) reduces to

$$\phi(\mathbf{r}) = \int \left[\phi(\mathbf{r}')\frac{\partial}{\partial n'}G(\mathbf{r};\mathbf{r}') - G(\mathbf{r};\mathbf{r}')\frac{\partial}{\partial n'}\phi(\mathbf{r}')\right]\cdot d\mathbf{S}'. \tag{10.66}$$

The potential at a field point $\mathbf{r}$ due to a point charge q located at the point $\mathbf{r}'$ is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\frac{q}{|\mathbf{r}-\mathbf{r}'|}.$$

Now

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

(the proof is left as an exercise for the reader) and it follows that the Green's function $G(\mathbf{r};\mathbf{r}')$in this case is equal

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{4\pi\varepsilon}\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

If the medium is bounded, the Green's function can be obtained by direct solution of Eq. (10.60) subject to the appropriate boundary conditions.

To illustrate the procedure of the Green's function technique, let us consider a simple example that can easily be solved by other methods. Consider two grounded parallel conducting plates of infinite extent: if the electric charge density ρ between the two plates is given, find the electric potential distribution ϕ between the plates. The electric potential distribution ϕ is described by solving Poisson's equation

$$\nabla^2\phi = -\rho/\varepsilon$$

subject to the boundary conditions

(1) $\phi(0) = 0$,
(2) $\phi(1) = 0$.

We take the coordinates shown in Fig. 10.3. Poisson's equation reduces to the simple form

$$\frac{d^2\phi}{dx^2} = -\frac{\rho}{\varepsilon}. \tag{10.67}$$

Instead of using the general result (10.64), it is more convenient to proceed directly. Multiplying Eq. (10.67) by $G(x;x')$ and integrating, we obtain

$$\int_0^1 G\frac{d^2\phi}{dx^2}dx = -\int_0^1 \frac{\rho(x)G}{\varepsilon}dx. \tag{10.68}$$

Then using integration by parts gives

$$\int_0^1 G\frac{d^2\phi}{dx^2}dx = G(x;x')\frac{d\phi(x)}{dx}\bigg|_0^1 - \int_0^1 \frac{dG}{dx}\frac{d\phi}{dx}dx$$

and using integration by parts again on the right hand side, we obtain

$$\begin{aligned}-\int_0^1 G\frac{d^2\phi}{dx^2}dx &= -G(x;x')\frac{d\phi(x)}{dx}\bigg|_0^1 + \left[\frac{dG}{dx}\phi\bigg|_0^1 - \int_0^1 \phi\frac{d^2G}{dx^2}dx\right] \\ &= G(0;x')\frac{d\phi(0)}{dx} - G(1;x')\frac{d\phi(1)}{dx} - \int_0^1 \phi\frac{d^2G}{dx^2}dx.\end{aligned}$$

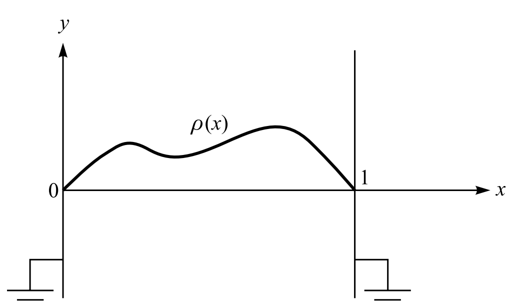

Figure 10.3.

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Substituting this into Eq. (10.68) we obtain

$$G(0;x')\frac{d\phi(0)}{dx} - G(1;x')\frac{d\phi(1)}{dx} - \int_0^1 \phi\,\frac{d^2G}{dx^2}\,dx = \int_0^1 \frac{G(x;x')\rho(x)}{\varepsilon}dx$$

or

$$\int_0^1 \phi\,\frac{d^2G}{dx^2}dx = G(1;x')\frac{d\phi(1)}{dx} - G(0;x')\frac{d\phi(0)}{dx} - \int_0^1 \frac{G(x;x')\rho(x)}{\varepsilon}dx. \quad (10.69)$$

We must now choose a Green's function which satisfies the following equation and the boundary conditions:

$$\frac{d^2G}{dx^2} = -\delta(x-x'), \quad G(0;x') = G(1;x') = 0. \quad (10.70)$$

Combining these with Eq. (10.69) we find the solution to be

$$\phi(x') = \int_0^1 \frac{1}{\varepsilon}\rho(x)G(x;x')dx. \quad (10.71)$$

It remains to find $G(x;x')$. By integration, we obtain from Eq. (10.70)

$$\frac{dG}{dx} = -\int \delta(x-x')dx + a = -U(x-x') + a,$$

where U is the unit step function and a is an integration constant to be determined later. Integrating once we get

$$G(x;x') = -\int U(x-x')dx + ax + b = -(x-x')U(x-x') + ax + b.$$

Imposing the boundary conditions on this general solution yields two equations:

$$G(0;x') = x'U(-x') + a\cdot 0 + b = 0 + 0 + b = 0,$$

$$G(1;x') = -(1-x')U(1-x') + a + b = 0.$$

From these we find

$$a = (1-x')U(1-x'), \quad b = 0$$

and the Green's function is

$$G(x;x') = -(x-x')U(x-x') + (1-x')x. \quad (10.72)$$

This gives the response at x' due to a unit source at x. Interchanging x and x' in Eqs. (10.70) and (10.71) we find the solution of Eq. (10.67) to be

$$\phi(x) = \int_0^1 \frac{1}{\varepsilon}\rho(x')G(x';x)dx' = \int_0^1 \frac{1}{\varepsilon}\rho(x')[-(x'-x)U(x'-x) + (1-x)x']dx'. \quad (10.73)$$

Note that the Green's function in the last equation can be written in the form

$$G(x;x') = \begin{cases} (1-x)x & x < x' \\ (1-x)x' & x > x' \end{cases}.$$

Laplace transform solutions of boundary-value problems

Laplace and Fourier transforms are useful in solving a variety of partial differential equations, the choice of the appropriate transforms depends on the type of boundary conditions imposed on the problem. To illustrate the use of the Lapace transforms in solving boundary-value problems, we solve the following equation:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \tag{10.74}$$

$$u(0,t) = u(3,t) = 0, \qquad u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x. \tag{10.75}$$

Taking the Laplace transform of Eq. (10.74) with respect to t gives

$$L\left[\frac{\partial u}{\partial t}\right] = 2L\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Now

$$L\left[\frac{\partial u}{\partial t}\right] = pL(u) - u(x,0)$$

and

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial x^2}dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-pt}u(x,t)dt = \frac{\partial^2}{\partial x^2}L[u].$$

Here $\partial^2/\partial x^2$ and $\int_0^\infty \cdots dt$ are interchangeable because x and t are independent. For convenience, let

$$U = U(x,p) = L[u(x,t)] = \int_0^\infty e^{-pt}u(x,t)dt.$$

We then have

$$pU - u(x,0) = 2\frac{d^2U}{dx^2},$$

from which we obtain, on using the given condition (10.75),

$$\frac{d^2U}{dx^2} - \frac{1}{2}pU = 3\sin 4\pi x - 5\sin 2\pi x. \tag{10.76}$$

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Now think of this as a differential equation in terms of x, with p as a parameter. Then taking the Laplace transform of the given conditions $u(0,t) = u(3,t) = 0$, we have

$$L[u(0,t)] = 0, \quad L[u(3,t)] = 0$$

or

$$U(0,p) = 0, \quad U(3,p) = 0.$$

These are the boundary conditions on $U(x,p)$. Solving Eq. (10.76) subject to these conditions we find

$$U(x,p) = \frac{5\sin 2\pi x}{p + 16\pi^2} - \frac{3\sin 4\pi x}{p + 64\pi^2}.$$

The solution to Eq. (10.74) can now be obtained by taking the inverse Laplace transform

$$u(x,t) = L^{-1}[U(x,p)] = 5e^{-16\pi^2 t}\sin 2\pi x - 3e^{-64\pi^2}\sin 4\pi x.$$

The Fourier transform method was used in Chapter 4 for the solution of ordinary linear ordinary differential equations with constant coefficients. It can be extended to solve a variety of partial differential equations. However, we shall not discuss this here. Also, there are other methods for the solution of linear partial differential equations. In general, it is a difficult task to solve partial differential equations analytically, and very often a numerical method is the best way of obtaining a solution that satisfies given boundary conditions.

Problems

10.1 (*a*) Show that $y(x,t) = F(2x + 5t) + G(2x - 5t)$ is a general solution of

$$4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}.$$

(*b*) Find a particular solution satisfying the conditions

$$y(0,t) = y(\pi,t) = 0, \quad y(x,0) = \sin 2x, \quad y'(x,0) = 0.$$

10.2. State the nature of each of the following equations (that is, whether elliptic, parabolic, or hyperbolic)

$$(a)\ \frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial^2 y}{\partial x^2} = 0, \qquad (b)\ x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + 3y^2\frac{\partial u}{\partial x}.$$

10.3 The electromagnetic wave equation: Classical electromagnetic theory was worked out experimentally in bits and pieces by Coulomb, Oersted, Ampere, Faraday and many others, but the man who put it all together and built it into the compact and consistent theory it is today was James Clerk Maxwell.

His work led to the understanding of electromagnetic radiation, of which light is a special case.

Given the four Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \qquad \text{(Gauss' law)},$$
$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \varepsilon_0 \partial \mathbf{E}/\partial t) \qquad \text{(Ampere's law)},$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss' law)},$$
$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad \text{(Faraday's law)},$$

where $\mathbf{B}$ is the magnetic induction, $\mathbf{j} = \rho \mathbf{v}$ is the current density, and μ_0 is the permeability of the medium, show that:

(*a*) the electric field and the magnetic induction can be expressed as

$$\mathbf{E} = -\nabla\phi - \partial \mathbf{A}/\partial t, \qquad \mathbf{B} = \nabla \times \mathbf{A},$$

where $\mathbf{A}$ is called the vector potential, and ϕ the scalar potential. It should be noted that $\mathbf{E}$ and $\mathbf{B}$ are invariant under the following transformations:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \phi' = \phi - \partial\phi/\partial t$$

in which χ is an arbitrary real function. That is, both $(\mathbf{A}', \phi)$, and $(\mathbf{A}', \phi')$ yield the same $\mathbf{E}$ and $\mathbf{B}$. Any condition which, for computational convenience, restricts the form of $\mathbf{A}$ and ϕ is said to define a gauge. Thus the above transformation is called a gauge transformation and χ is called a gauge parameter.

(*b*) If we impose the so-called Lorentz gauge condition on $\mathbf{A}$ and ϕ:

$$\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0(\partial\phi/\partial t) = 0,$$

then both $\mathbf{A}$ and ϕ satisfy the following wave equations:

$$\nabla^2\mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j},$$

$$\nabla^2\phi - \mu_0\varepsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0.$$

10.4 Given Gauss' law $\iint_S \mathbf{E} \cdot d\mathbf{s} = q/\varepsilon$, find the electric field produced by a charged plane of infinite extension is given by $E = \sigma/\varepsilon$, where σ is the charge per unit area of the plane.

10.5 Consider an infinitely long uncharged conducting cylinder of radius l placed in an originally uniform electric field E_0 directed at right angles to the axis of the cylinder. Find the potential at a point $\rho(> l)$ from the axis of the cylinder. The boundary conditions are:

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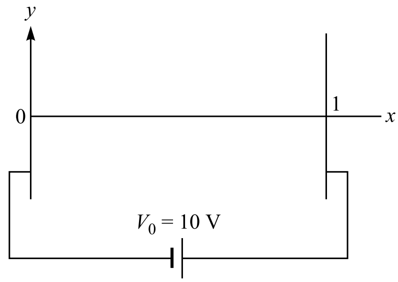

Figure 10.4.

$$\phi(\rho, \varphi) = \begin{cases} -E_0\rho \cos \varphi = -E_0 x & \text{for} \quad \rho \to \infty, \\ 0 & \text{for} \quad \rho = l, \end{cases}$$

where the x-axis has been taken in the direction of the uniform field E_0.

10.6 Obtain the solution of the heat conduction equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t}$$

which satisfies the boundary conditions
(1) $u(0,t) = u(l,t) = 0,\ t \geq 0,$ (2) $u(x,0) = f(x), 0 \leq x,$ where $f(x)$ is a given function and l is a constant.

10.7 If a battery is connected to the plates as shown in Fig. 10.4, and if the charge density distribution between the two plates is still given by $\rho(x)$, find the potential distribution between the plates.

10.8 Find the Green's function that satisfies the equation

$$\frac{d^2G}{dx^2} = \delta(x - x')$$

and the boundary conditions $G = 0$ when $x = 0$ and G remains bounded when x approaches infinity. (This Green's function is the potential due to a surface charge $-\varepsilon$ per unit area on a plane of infinite extent located at $x = x'$ in a dielectric medium of permittivity ε when a grounded conducting plane of infinite extent is located at $x = 0$.)

10.9 Solve by Laplace transforms the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t} \quad \text{for} \quad x > 0,\ t > 0,$$

given that $u = u_0$ (a constant) on $x = 0$ for $t > 0$, and $u = 0$ for $x > 0, t = 0$.