11

Simple linear integral equations

In previous chapters we have met equations in which the unknown functions appear under an integral sign. Such equations are called integral equations. Fourier and Laplace transforms are important integral equations, In Chapter 4, by introducing the method of Green's function we were led in a natural way to reformulate the problem in terms of integral equations. Integral equations have become one of the very useful and sometimes indispensable mathematical tools of theoretical physics and engineering.

Classification of linear integral equations

In this chapter we shall confine our attention to linear integral equations. Linear integral equations can be divided into two major groups:

(1) If the unknown function occurs only under the integral sign, the integral equation is said to be of the first kind. Integral equations having the unknown function both inside and outside the integral sign are of the second kind.
(2) If the limits of integration are constants, the equation is called a Fredholm integral equation. If one limit is variable, it is a Volterra equation.

These four kinds of linear integral equations can be written as follows:

$$f(x) = \int_a^b K(x,t)u(t)dt \qquad \text{Fredholm equation of the first kind;} \tag{11.1}$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad \text{Fredholm equation of the second kind;} \tag{11.2}$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

$$f(x) = \int_a^x K(x,t)u(t)dt \qquad \text{Volterra equation of the first kind;} \tag{11.3}$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \quad \text{Volterra equation of the second kind.} \tag{11.4}$$

In each case $u(t)$ is the unknown function, $K(x,t)$ and $f(x)$ are assumed to be known. $K(x,t)$ is called the kernel or nucleus of the integral equation. λ is a parameter, which often plays the role of an eigenvalue. The equation is said to be homogeneous if $f(x) = 0$.

If one or both of the limits of integration are infinite, or the kernel $K(x,t)$ becomes infinite in the range of integration, the equation is said to be singular; special techniques are required for its solution.

The general linear integral equation may be written as

$$h(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{11.5}$$

If $h(x) = 0$, we have a Fredholm equation of the first kind; if $h(x) = 1$, we have a Fredholm equation of the second kind. We have a Volterra equation when the upper limit is x.

It is beyond the scope of this book to present the purely mathematical general theory of these various types of equations. After a general discussion of a few methods of solution, we will illustrate them with some simple examples. We will then show with a few examples from physical problems how to convert differential equations into integral equations.

Some methods of solution

Separable kernel

When the two variables x and t which appear in the kernel $K(x,t)$ are separable, the problem of solving a Fredholm equation can be reduced to that of solving a system of algebraic equations, a much easier task. When the kernel $K(x,t)$ can be written as

$$K(x,t) = \sum_{i=1}^{n} g_i(x)h_i(t), \tag{11.6}$$

where $g(x)$ is a function of x only and $h(t)$ a function of t only, it is said to be degenerate. Putting Eq. (11.6) into Eq. (11.2), we obtain

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} \int_a^b g_i(x)h_i(t)u(t)dt.$$

Note that $g(x)$ is a constant as far as the t integration is concerned, hence it may be taken outside the integral sign and we have

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} g_i(x) \int_a^b h_i(t)u(t)dt. \tag{11.7}$$

Now

$$\int_a^b h_i(t)u(t)dt = C_i \ (= \text{const.}). \tag{11.8}$$

Substituting this into Eq. (11.7) and solving for $u(t)$, we obtain

$$u(t) = f(x) + \lambda C \sum_{i=1}^{n} g_i(x). \tag{11.9}$$

The value of C_i may now be obtained by substituting Eq. (11.9) into Eq. (11.8). The solution is only valid for certain values of λ, and we call these the eigenvalues of the integral equation. The homogeneous equation has non-trivial solutions only if λ is one of these eigenvalues; these solutions are called eigenfunctions of the kernel (operator) K.

Example 11.1
As an example of this method, we consider the following equation:

$$u(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)u(t)dt. \tag{11.10}$$

This is a Fredholm equation of the second kind, with $f(x) = x$ and $K(x,t) = xt^2 + x^2 t$. If we define

$$\alpha = \int_0^1 t^2 u(t)dt, \quad \beta = \int_0^1 tu(t)dt, \tag{11.11}$$

then Eq. (11.10) becomes

$$u(x) = x + \lambda(\alpha x + \beta x^2). \tag{11.12}$$

To determine A and B, we put Eq. (11.12) back into Eq. (11.11) and obtain

$$\alpha = \tfrac{1}{4} + \tfrac{1}{4}\lambda\alpha + \tfrac{1}{5}\lambda\beta, \quad \beta = \tfrac{1}{3} + \tfrac{1}{3}\lambda\alpha + \tfrac{1}{4}\lambda\beta. \tag{11.13}$$

Solving this for α and β we find

$$\alpha = \frac{60 + \lambda}{240 - 120\lambda - \lambda^2}, \quad \beta = \frac{80}{240 - 120\lambda - \lambda^2},$$

and the final solution is

$$u(t) = \frac{(240 - 60\lambda)x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}.$$

The solution blows up when $\lambda = 117.96$ or $\lambda = 2.04$. These are the eigenvalues of the integral equation.

Fredholm found that if: (1) $f(x)$ is continuous, (2) $K(x, t)$ is piecewise continuous, (3) the integrals $\iint K^2(x, t)dxdt$, $\int f^2(t)dt$ exist, and (4) the integrals $\iint K^2(x, t)dt$ and $\int K^2(t, x)dt$ are bounded, then the following theorems apply:

(a) Either the inhomogeneous equation

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

has a unique solution for any function $f(x)$(λ is not an eigenvalue), or the homogeneous equation

$$u(x) = \lambda \int_a^b K(x, t)u(t)dt$$

has at least one non-trivial solution corresponding to a particular value of λ. In this case, λ is an eigenvalue and the solution is an eigenfunction.

(b) If λ is an eigenvalue, then λ is also an eigenvalue of the transposed equation

$$u(x) = \lambda \int_a^b K(t, x)u(t)dt;$$

and, if λ is not an eigenvalue, then λ is also not an eigenvalue of the transposed equation

$$u(x) = f(x) + \lambda \int_a^b K(t, x)u(t)dt.$$

(c) If λ is an eigenvalue, the inhomogeneous equation has a solution if, and only if,

$$\int_a^b u(x)f(x)dx = 0$$

for every function $f(x)$.

We refer the readers who are interested in the proof of these theorems to the book by R. Courant and D. Hilbert (*Methods of Mathematical Physics*, Vol. 1, Wiley, 1961).

Neumann series solutions

This method is due largely to Neumann, Liouville, and Volterra. In this method we solve the Fredholm equation (11.2)

$$u(x) = f(x) + \lambda \int_a^b K(x, t)u(t)dt$$

by iteration or successive approximations, and begin with the approximation

$$u(x) \approx u_0(x) \approx f(x).$$

This approximation is equivalent to saying that the constant λ or the integral is small. We then put this crude choice into the integral equation (11.2) under the integral sign to obtain a second approximation:

$$u_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt$$

and the process is then repeated and we obtain

$$u_2(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t') f(t') dt' dt.$$

We can continue iterating this process, and the resulting series is known as the Neumann series, or Neumann solution:

$$u(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b \int_a^b K(x,t) K(t,t') f(t') dt' dt + \cdots.$$

This series can be written formally as

$$u_n(x) = \sum_{i=1}^{n} \lambda^i \varphi_i(x), \tag{11.14}$$

where

$$\left.\begin{aligned}
\varphi_0(x) &= u_0(x) = f(x),\\
\varphi_1(x) &= \int_a^b K(x,t_1) f(t_1) dt_1,\\
\varphi_2(x) &= \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) f(t_2) dt_1 dt_2,\\
&\vdots\\
\varphi_n(x) &= \int_a^b \int_a^b \cdots \int_a^b K(x,t_1) K(t_1,t_2) \cdots K(t_{n-1},t_n) f(t_n) dt_1 dt_2 \cdots dt_n.
\end{aligned}\right\} \tag{11.15}$$

The series (11.14) will converge for sufficiently small λ, when the kernel $K(x,t)$ is bounded. This can be checked with the Cauchy ratio test (Problem 11.4).

Example 11.2
Use the Neumann method to solve the integral equation

$$u(x) = f(x) + \tfrac{1}{2} \int_{-1}^{1} K(x,t) u(t) dt, \tag{11.16}$$

where

$$f(x) = x, \qquad K(x,t) = t - x.$$

Solution: We begin with

$$u_0(x) = f(x) = x.$$

Then

$$u_1(x) = x + \frac{1}{2}\int_{-1}^{1}(t - x)t\,dt = x + \frac{1}{3}.$$

Putting $u_1(x)$ into Eq. (11.16) under the integral sign, we obtain

$$u_2(x) = x + \frac{1}{2}\int_{-1}^{1}(t - x)\left(t + \frac{1}{3}\right)dt = x + \frac{1}{3} - \frac{x}{3}.$$

Repeating this process of substituting back into Eq. (11.16) once more, we obtain

$$u_3(x) = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2}.$$

We can improve the approximation by iterating the process, and the convergence of the resulting series (solution) can be checked out with the ratio test.

The Neumann method is also applicable to the Volterra equation, as shown by the following example.

Example 11.3

Use the Neumann method to solve the Volterra equation

$$u(x) = 1 + \lambda\int_0^x u(t)dt.$$

Solution: We begin with the zeroth approximation $u_0(x) = 1$. Then

$$u_1(x) = 1 + \lambda\int_0^x u_0(t)dt = 1 + \lambda\int_0^x dt = 1 + \lambda x.$$

This gives

$$u_2(x) = 1 + \lambda\int_0^x u_1(t)dt = 1 + \lambda\int_0^x (1 + \lambda t)dt = 1 + \lambda x + \frac{1}{2}\lambda^2 x^2;$$

similarly,

$$u_3(x) = 1 + \lambda\int_0^x \left(1 + \lambda t + \frac{1}{2}\lambda^2 t^2\right)dt = 1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \frac{1}{3!}\lambda^3 x^3.$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

By induction

$$u_n(x) = \sum_{k=1}^{n} \frac{1}{k!} \lambda^k x^k.$$

When $n \to \infty$, $u_n(x)$ approaches

$$u(x) = e^{\lambda x}.$$

Transformation of an integral equation into a differential equation

Sometimes the Volterra integral equation can be transformed into an ordinary differential equation which may be easier to solve than the original integral equation, as shown by the following example.

Example 11.4

Consider the Volterra integral equation $u(x) = 2x + 4\int_0^x (t - x)u(t)dt$. Before we transform it into a differential equation, let us recall the following very useful formula: if

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)dx,$$

where a and b are continuous and at least once differentiable functions of α, then

$$\frac{dI(\alpha)}{d\alpha} = f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx.$$

With the help of this formula, we obtain

$$\begin{aligned} \frac{d}{dx}u(x) &= 2 + 4\left[\{(t - x)u(t)\}_{t=x} - \int_0^x u(t)dt\right] \\ &= 2 - 4\int_0^x u(t)dt. \end{aligned}$$

Differentiating again we obtain

$$\frac{d^2u(x)}{dx^2} = -4u(x).$$

This is a differentiation equation equivalent to the original integral equation, but its solution is much easier to find:

$$u(x) = A\cos 2x + B\sin 2x,$$

where A and B are integration constants. To determine their values, we put the solution back into the original integral equation under the integral sign, and then integration gives $A = 0$ and $B = 1$. Thus the solution of the original integral equation is

$$u(x) = \sin 2x.$$

Laplace transform solution

The Volterra integral equation can sometime be solved with the help of the Laplace transformation and the convolution theorem. Before we consider the Laplace transform solution, let us review the convolution theorem. If $f_1(x)$ and $f_2(x)$ are two arbitrary functions, we define their convolution (*faltung* in German) to be

$$g(x) = \int_{-\infty}^{\infty} f_1(y) f_2(x - y) dy.$$

Its Laplace transform is

$$L[g(x)] = L[f_1(x)]L[f_2(x)].$$

We now consider the Volterra equation

$$\begin{aligned} u(x) &= f(x) + \lambda \int_0^x K(x, t) u(t) dt \\ &= f(x) + \lambda \int_0^x g(x - t) u(t) dt, \end{aligned} \tag{11.17}$$

where $K(x - t) = g(x - t)$, a so-called displacement kernel. Taking the Laplace transformation and using the convolution theorem, we obtain

$$L\left[\int_0^x g(x - t) u(t) dt\right] = L[g(x - t)]L[u(t)] = G(p)U(p),$$

where $U(p) = L[u(t)] = \int_0^\infty e^{-pt} u(t) dt$, and similarly for $G(p)$. Thus, taking the Laplace transformation of Eq. (11.17), we obtain

$$U(p) = F(p) + \lambda G(p) U(p)$$

or

$$U(p) = \frac{F(p)}{1 - \lambda G(p)}.$$

Inverting this we obtain $u(t)$:

$$u(t) = L^{-1}\left[\frac{F(p)}{1 - \lambda G(p)}\right].$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

Fourier transform solution

If the kernel is a displacement kernel and if the limits are $-\infty$ and $+\infty$, we can use Fourier transforms. Consider a Fredholm equation of the second kind

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x-t)u(t)dt. \tag{11.18}$$

Taking Fourier transforms (indicated by overbars)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ipx} = \bar{f}(p), \text{etc.},$$

and using the convolution theorem

$$\int_{-\infty}^{\infty} f(t) g(x-t) dt = \int_{-\infty}^{\infty} \bar{f}(y) \bar{g}(y) e^{-iyx} dy,$$

we obtain the transform of our integral equation (11.18):

$$\bar{u}(p) = \bar{f}(p) + \lambda \bar{K}(p) \bar{u}(p).$$

Solving for $\bar{u}(p)$ we obtain

$$\bar{u}(p) = \frac{\bar{f}(p)}{1 - \lambda \bar{K}(p)}.$$

If we can invert this equation, we can solve the original integral equation:

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(t) e^{-ixt}}{1 - \sqrt{2\pi}\lambda K(t)}. \tag{11.19}$$

The Schmidt–Hilbert method of solution

In many physical problems, the kernel may be symmetric. In such cases, the integral equation may be solved by a method quite different from any of those in the preceding section. This method, devised by Schmidt and Hilbert, is based on considering the eigenfunctions and eigenvalues of the homogeneous integral equation.

A kernel $K(x,t)$ is said to be symmetric if $K(x,t) = K(t,x)$ and Hermitian if $K(x,y) = K^*(t,x)$. We shall limit our discussion to such kernels.

(a) The homogeneous Fredholm equation

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt.$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

A Hermitian kernel has at least one eigenvalue and it may have an infinite number. The proof will be omitted and we refer interested readers to the book by Courant and Hibert mentioned earlier (Chapter 3).

The eigenvalues of a Hermitian kernel are real, and eigenfunctions belonging to different eigenvalues are orthogonal; two functions $f(x)$ and $g(x)$ are said to be orthogonal if

$$\int f^*(x)g(x)dx = 0.$$

To prove the reality of the eigenvalue, we multiply the homogeneous Fredholm equation by $u^*(x)$, then integrating with respect to x, we obtain

$$\int_a^b u^*(x)u(x)dx = \lambda \int_a^b \int_a^b K(x,t)u^*(x)u(t)dtdx. \tag{11.20}$$

Now, multiplying the complex conjugate of the Fredholm equation by $u(x)$ and then integrating with respect to x, we get

$$\int_a^b u^*(x)u(x)dx = \lambda^* \int_a^b \int_a^b K^*(x,t)u^*(t)u(x)dtdx.$$

Interchanging x and t on the right hand side of the last equation and remembering that the kernel is Hermitian $K^*(t,x) = K(x,t)$, we obtain

$$\int_a^b u^*(x)u(x)dx = \lambda^* \int_a^b \int_a^b K(x,t)u(t)u^*(x)dtdx.$$

Comparing this equation with Eq. (11.2), we see that $\lambda = \lambda^*$, that is, λ is real.

We now prove the orthogonality. Let λ_i, λ_j be two different eigenvalues and $u_i(x), u_j(x)$, the corresponding eigenfunctions. Then we have

$$u_i(x) = \lambda_i \int_a^b K(x,t)u_i(t)dt, \qquad u_j(x) = \lambda_j \int_a^b K(x,t)u_j(t)dt.$$

Now multiplying the first equation by $\lambda u_j(x)$, the second by $\lambda_i u_i(x)$, and then integrating with respect to x, we obtain

$$\begin{aligned} \lambda_j \int_a^b u_i(x)u_j(x)dx &= \lambda_i\lambda_j \int_a^b \int_a^b K(x,t)u_i(t)u_j(x)dtdx, \\ \lambda_i \int_a^b u_i(x)u_j(x)dx &= \lambda_i\lambda_j \int_a^b \int_a^b K(x,t)u_j(t)u_i(x)dtdx. \end{aligned} \tag{11.21}$$

Now we interchange x and t on the right hand side of the last integral and because of the symmetry of the kernel, we have

$$\lambda_i \int_a^b u_i(x)u_j(x)dx = \lambda_i\lambda_j \int_a^b \int_a^b K(x,t)u_i(t)u_j(x)dtdx. \tag{11.22}$$

Subtracting Eq. (11.21) from Eq. (11.22), we obtain

$$(\lambda_i - \lambda_j) \int_a^b u_i(x) u_j(x) dx = 0. \tag{11.23}$$

Since $\lambda_i \neq \lambda_j$, it follows that

$$\int_a^b u_i(x) u_j(x) dx = 0. \tag{11.24}$$

Such functions may always be nomalized. We will assume that this has been done and so the solutions of the homogeneous Fredholm equation form a complete orthonomal set:

$$\int_a^b u_i(x) u_j(x) dx = \delta_{ij}. \tag{11.25}$$

Arbitrary functions of x, including the kernel for fixed t, may be expanded in terms of the eigenfunctions

$$K(x,t) = \sum C_i u_i(x). \tag{11.26}$$

Now substituting Eq. (11.26) into the original Fredholm equation, we have

$$\begin{aligned} u_j(t) &= \lambda_j \int_a^b K(t,x) u_j(x) dx = \lambda_j \int_a^b K(x,t) u_j(x) dx \\ &= \lambda_j \sum_i \int_a^b C_i u_i(x) u_j(x) dx = \lambda_j \sum_i C_i \delta_{ij} = \lambda_j C_j \end{aligned}$$

or

$$C_i = u_i(t)/\lambda_i$$

and for our homogeneous Fredholm equation of the second kind the kernel may be expressed in terms of the eigenfunctions and eigenvalues as

$$K(x,t) = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(t)}{\lambda_n}. \tag{11.27}$$

The Schmidt–Hilbert theory does not solve the homogeneous integral equation; its main function is to establish the properties of the eigenvalues (reality) and eigenfunctions (orthogonality and completeness). The solutions of the homogeneous integral equation come from the preceding section on methods of solution.

(b) Solution of the inhomogeneous equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t) u(t) dt. \tag{11.28}$$

We assume that we have found the eigenfunctions of the homogeneous equation by the methods of the preceding section, and we denote them by $u_i(x)$. We may now expand both $u(x)$ and $f(x)$ in terms of $u_i(x)$, which forms an orthonormal complete set.

$$u(x) = \sum_{n=1}^{\infty} \alpha_n u_n(x), \quad f(x) = \sum_{n=1}^{\infty} \beta_n u_n(x). \tag{11.29}$$

Substituting Eq. (11.29) into Eq. (11.28), we obtain

$$\begin{aligned}
\sum_{n=1}^{n} \alpha_n u_n(x) &= \sum_{n=1}^{n} \beta_n u_n(x) + \lambda \int_a^b K(x,t) \sum_{n=1}^{n} \alpha_n u_n(t) dt \\
&= \sum_{n=1}^{n} \beta_n u_n(x) + \lambda \sum_{n=1}^{\infty} \alpha_n \frac{u_m(x)}{\lambda_m} \int_a^b u_m(t) u_n(t) dt, \\
&= \sum_{n=1}^{n} \beta_n u_n(x) + \lambda \sum_{n=1}^{\infty} \alpha_n \frac{u_m(x)}{\lambda_m} \delta_{nm},
\end{aligned}$$

from which it follows that

$$\sum_{n=1}^{n} \alpha_n u_n(x) = \sum_{n=1}^{n} \beta_n u_n(x) + \lambda \sum_{n=1}^{\infty} \frac{\alpha_n u_n(x)}{\lambda_n}. \tag{11.30}$$

Multiplying by $u_i(x)$ and then integrating with respect to x from a to b, we obtain

$$\alpha_n = \beta_n + \lambda \alpha_n / \lambda_n, \tag{11.31}$$

which can be solved for α_n in terms of β_n:

$$\alpha_n = \frac{\lambda_n}{\lambda_n - \lambda} \beta_n, \tag{11.32}$$

where β_n is given by

$$\beta_n = \int_a^b f(t) u_n(t) dt. \tag{11.33}$$

Finally, our solution is given by

$$\begin{aligned}
u(x) &= f(x) + \lambda \sum_{n=1}^{\infty} \frac{\alpha_n u_n(x)}{\lambda_n} \\
&= f(x) + \lambda \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n - \lambda} u_n(x),
\end{aligned} \tag{11.34}$$

where β_n is given by Eq. (11.33), and $\lambda_i \neq \lambda$.

When λ for the inhomogeneous equation is equal to one of the eigenvalues, λ_k, of the kernel, our solution (11.31) blows up. Let us return to Eq. (11.31) and see what happens to α_k:

$$\alpha_k = \beta_k + \lambda_k \alpha_k / \lambda_k = \beta_k + \alpha_k.$$

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Clearly, $\beta_k = 0$, and α_k is no longer determined by β_k. But we have, according to Eq. (11.33),

$$\int_a^b f(t)u_k(t)dt = \beta_k = 0, \tag{11.35}$$

that is, $f(x)$ is orthogonal to the eigenfunction $u_k(x)$. Thus if $\lambda = \lambda_k$, the inhomogeneous equation has a solution only if $f(x)$ is orthogonal to the corresponding eigenfunction $u_k(x)$. The general solution of the equation is then

$$u(x) = f(x) + \alpha_k u_k(x) + \alpha_k \sum_{n=1'}^{\infty}{}' \frac{\int_a^b f(t)u_n(t)dt}{\lambda_n - \lambda_k} u_n(x), \tag{11.36}$$

where the prime on the summation sign means that the term $n = k$ is to be omitted from the sum. In Eq. (11.36) the α_k remains as an undetermined constant.

Relation between differential and integral equations

We have shown how an integral equation can be transformed into a differential equation that may be easier to solve than the original integral equation. We now show how to transform a differential equation into an integral equation. After we become familiar with the relation between differential and integral equations, we may state the physical problem in either form at will. Let us consider a linear second-order differential equation

$$x'' + A(t)x' + B(t)x = g(t), \tag{11.37}$$

with the initial condition

$$x(a) = x_0, \quad x'(a) = x_0'.$$

Integrating Eq. (11.37), we obtain

$$x' = -\int_a^t Ax'dt - \int_a^t Bxdt + \int_a^t gdt + C_1.$$

The initial conditions require that $C_1 = x_0'$. We next integrate the first integral on the right hand side by parts and obtain

$$x' = -Ax - \int_a^t (B - A')xdt + \int_a^t gdt + A(a)x_0 + x_0'.$$

Integrating again, we get

$$x = -\int_a^t Axdt - \int_a^t \int_a^t \left[B(y) - A'(y)\right]x(y)dydt$$
$$+ \int_a^t \int_a^t g(y)dydt + \left[A(a)x_0 + x_0'\right](t - a) + x_0.$$

Then using the relation

$$\int_a^t \int_a^t f(y)dydt = \int_a^t (t-y)f(y)dy,$$

we can rewrite the last equation as

$$x(t) = -\int_a^t \left[A(y) + (t-y)\left\{B(y) - A'(y)\right\}\right]x(y)dy$$
$$+ \int_a^t (t-y)g(y)dy + \left[A(a)x_0 + x_0'\right](t-a) + x_0, \tag{11.38}$$

which can be put into the form of a Volterra equation of the second kind

$$x(t) = f(t) + \int_a^t K(t,y)x(y)dy, \tag{11.39}$$

with

$$K(t,y) = (y-t)[B(y) - A'(y)] - A(y), \tag{11.39a}$$

$$f(t) = \int_0^t (t-y)g(y)dy + [A(a)x_0 + x_0'](t-a) + x_0. \tag{11.39b}$$

Use of integral equations

We have learned how linear integral equations of the more common types may be solved. We now show some uses of integral equations in physics; that is, we are going to state some physical problems in integral equation form. In 1823, Abel made one of the earliest applications of integral equations to a physical problem. Let us take a brief look at this old problem in mechanics.

Abel's integral equation

Consider a particle of mass m falling along a smooth curve in a vertical plane, the yz plane, under the influence of gravity, which acts in the negative z direction. Conservation of energy gives

$$\frac{1}{2}m(\dot{z}^2 + \dot{y}^2) + mgz = E,$$

where $\dot{z} = dz/dt$, and $\dot{y} = dy/dt$. If the shape of the curve is given by $y = F(z)$, we can write $\dot{y} = (dF/dz)\dot{z}$. Substituting this into the energy conservation equation and solving for $\dot{z}$, we obtain

$$\dot{z} = \frac{\sqrt{2E/m - 2gz}}{\sqrt{1 + (dF/dz)^2}} = \frac{\sqrt{E/mg - z}}{u(z)}, \tag{11.40}$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

where

$$u(z) = \sqrt{1 + (dF/dz)^2/2g}.$$

If $\dot{z} = 0$ and $z = z_0$ at $t = 0$, then $E/mg = z_0$ and Eq. (11.40) becomes

$$\dot{z} = \sqrt{z_0 - z}/u(z).$$

Solving for time t, we obtain

$$t = -\int_z^{z_0} \frac{u(z)}{\sqrt{z_0 - z}}\, dz = \int_{z_0}^{z} \frac{u(z)}{\sqrt{z_0 - z}}\, dz,$$

where z is the height the particle reaches at time t.

Classical simple harmonic oscillator

Consider a linear oscillator

$$\ddot{x} + \omega^2 x = 0, \quad \text{with} \quad x(0) = 0, \quad \dot{x}(0) = 1.$$

We can transform this differential equation into an integral equation. Comparing with Eq. (11.37), we have

$$A(t) = 0, \quad B(t) = \omega^2, \quad \text{and} \quad g(t) = 0.$$

Substituting these into Eq. (11.38) (or (11.39), (11.39a), and (11.39b)), we obtain the integral equation

$$x(t) = t + \omega^2 \int_0^t (y - t)x(y)dy,$$

which is equivalent to the original differential equation plus the initial conditions.

Quantum simple harmonic oscillator

The Schrödinger equation for the energy eigenstates of the one-dimensional simple harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi. \tag{11.41}$$

Changing to the dimensionless variable $y = \sqrt{m\omega/\hbar}x$, Eq. (11.41) reduces to a simpler form:

$$\frac{d^2\psi}{dy^2} + (\alpha^2 - y^2)\psi = 0, \tag{11.42}$$

where $\alpha = \sqrt{2E/\hbar\omega}$. Taking the Fourier transform of Eq. (11.42), we obtain

$$\frac{d^2 g(k)}{dk^2} + (\alpha^2 - k^2)g(k) = 0, \tag{11.43}$$

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

where

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(y) e^{iky} dy \tag{11.44}$$

and we also assume that ψ and ψ' vanish as $y \to \pm\infty$.

Eq. (11.43) is formally identical to Eq. (11.42). Since quantities such as the total probability and the expectation value of the potential energy must be remain finite for finite E, we should expect $g(k), dg(k)/dk \to 0$ as $k \to \pm\infty$. Thus g and ψ differ at most by a normalization constant

$$g(k) = c\psi(k).$$

It follows that ψ satisfies the integral equation

$$c\psi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(y) e^{iky} dy. \tag{11.45}$$

The constant c may be determined by substituting $c\psi$ on the right hand side:

$$\begin{aligned} c^2\psi(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(z) e^{izy} e^{iky} dz dy \\ &= \int_{-\infty}^{\infty} \psi(z) \delta(z+k) dz \\ &= \psi(-k). \end{aligned}$$

Recall that ψ may be simultaneously chosen to be a parity eigenstate $\psi(-x) = \pm\psi(x)$. We see that eigenstates of even parity require $c^2 = 1$, or $c = \pm 1$; and for eigenstates of odd parity we have $c^2 = -1$, or $c = \pm i$.

We shall leave the solution of Eq. (11.45), which can be approached in several ways, as an exercise for the reader.

Problems

11.1 Solve the following integral equations:

(*a*) $u(x) = \frac{1}{2} - x + \int_0^1 u(t) dt;$

(*b*) $u(x) = \lambda \int_0^1 u(t) dt;$

(*c*) $u(x) = x + \lambda \int_0^1 u(t) dt.$

11.2 Solve the Fredholm equation of the second kind

$$f(x) = u(x) + \lambda \int_a^b K(x,t) u(t) dt,$$

where $f(x) = \cosh x, K(x,t) = xt$.

https://doi.org/10.1017/CBO9780511755781.012 Published online by Cambridge University Press

11.3 The homogeneous Fredholm equation

$$u(x) = \lambda \int_0^{\pi/2} \sin x \sin t u(t) dt$$

only has a solution for a particular value of λ. Find the value of λ and the solution corresponding to this value of λ.

11.4 Solve homogeneous Fredholm equation $u(x) = \lambda \int_{-1}^{1} (t + x) u(t) dt$. Find the values of λ and the corresponding solutions.

11.5 Check the convergence of the Neumann series (11.14) by the Cauchy ratio test.

11.6 Transform the following differential equations into integral equations:

$$(a)\ \frac{dx}{dt} - x = 0 \quad \text{with} \quad x = 1 \text{ when } t = 0;$$

$$(b)\ \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 1 \quad \text{with} \quad x = 0,\ \frac{dx}{dt} = 1 \text{ when } t = 0.$$

11.7 By using the Laplace transformation and the convolution theorem solve the equation

$$u(x) = x + \int_0^x \sin(x - t) u(t) dt.$$

11.8 Given the Fredholm integral equation

$$e^{-x^2} = \int_{-\infty}^{\infty} e^{-(x-t)^2} u(t) dt,$$

apply the Fouurier convolution technique to solve it for $u(t)$.

11.9 Find the solution of the Fredholm equation

$$u(x) = x + \lambda \int_0^1 (x + t) u(t) dt$$

by the Schmidt–Hilbert method for λ not equal to an eigenvalue. Show that there are no solutions when λ is an eigenvalue.